Sapienza Università di Roma

Corso di Laurea Specialistica in Fisica

Higgs decay into two photons: a recent discussion

Tesi di Laurea Specialistica in Fisica

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Sessione Estiva

Anno Accademico 2011-2012
Dipartimento di Fisica

– Your thesis that the Higgs boson is a black hole accelerating backwards through time is fascinating.
– Thank you. It just– it came to me one morning in the shower.
– That's nice. Too bad it's wrong.

Stephen Hawking and Sheldon Cooper

Abstract

We reanalyze the recent computation of the amplitude of the Higgs boson decay into two photons presented by Gastmans *et al.* [1, 2]. The reasons for which this result cannot be the correct one have been discussed in some recent papers. We address here the general issue of the indeterminacy of integrals with four-dimensional gauge-breaking regulators and to which extent it might eventually be solved by imposing physical constraints. Imposing gauge invariance as the last step upon R_ξ-gauge calculations with four-dimensional gauge-breaking regulators, to recover the well known $H \to \gamma\gamma$ result is indeed allowed. However we show that in the particular case of the unitary gauge, the indeterminacy cannot be tackled in this same way. The combination of unitary gauge with a cutoff regularization scheme turns out to be non-predictive.

Contents

Chapter 1

Introduction

Last summer some attention has been brought back to the W-loop contribution in the calculation of the $H \to \gamma\gamma$ amplitude because of a result presented by Gastmans *et al.* [1,2] turning out to be at odds with the renowned one of Refs. [3,4]. It goes without saying that, if correct, the result in Refs. [1, 2] would have had important consequences for high energy physics. Firstly, the decay width of the Higgs boson to two photons would have been halved with respect to the old result; therefore, the search for the Higgs in this channel would have needed much more data (and time, and money) than expected, and thousands of rows of code should have been rewritten to take into account the new result. And all this would have happened to one of the most promising channels for the search of a low-mass Higgs at LHC. Moreover, Gastmans *et al.* held dimensional regularization used in [3] responsible for the failure of the renowned result; and so, the most used regularization technique for non-anomalous theories would have been challenged, and some general questions about the significance of regularization in general would have been answered.

For the peace of everybody, the new result is wrong: the decoupling theorem [5] used to refute the old one has been misapplied, whereas Gastmans *et al.*'s result does not fulfill the Goldstone bosons Equivalence theorem. Indeed, in the months after the publication of Gastmans *et al.*'s article, the scientific community reacted (and sometimes over-reacted): a few papers [6–13] appeared on the arXiv critizing the new result, and presenting new calculations leading to the renowned one. This thesis was started just in the middle of this "race" to see who confuted Gastmans *et al.* first. However, once shown that Gastmans *et al.* are wrong, it is still interesting to understand where the weak spot in the calculation procedure followed by Gastmans *et al.* is. Moreover, Shao *et al.* [9] confirmed that in unitarity gauge, the use of a cutoff regularization leads in any case to Gastmans *et al.*'s result.

In this thesis we try to understand the significance of regularization of ill-defined integrals, and how different regulators can lead to different results, introducing an indeterminacy in the calculations. In Chapter 2 we introduce the Higgs mechanism, focusing on the definition of the unitary gauge. In Chapter 3 we briefly review Gastmans *et al.*'s paper, showing the crucial spot where the problem arises. In Chapter 4 we introduce the Goldstone bosons Equivalence theorem and the decoupling theorem, and show why Gastmans *et al.*'s result does not fulfill the former, and misapplies the latter. In Chapter 5 we introduce cutoff regularization, and show that it leads to different results in different gauges. In Chapter 6 we explain how the use of regularizations which do not mantain the full symmetry of the theory can lead to indeterminate results, and that the indeterminacy cannot be resolved *a posteriori* in unitary gauge, so that the result by Gastmans *et al.* is to be intended up to constants. Our conclusions are drawn in Chapter 7.

Chapter 2

The Higgs mechanism

As a first step, we briefly describe the history of the Higgs mechanism, and the major features of spontaneously broken gauge theories. In the Fifties, the Fermi theory of current-current interactions received some improvements by the works of Feynman and Gell-Mann, leading to the renowned $V-A$ interaction. However, it was clear that such a theory could not be the final one, because of the violation of unitarity above 300 GeV. In 1961, Glashow [14] proposed a Yang-Mills gauge theory $SU(2)_L \otimes U(1)_Y$ to unify both weak and electromagnetic lepton interactions; for a massless chiral lagrangian, we have

$$\mathcal{L} = -\frac{1}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{2} \operatorname{tr} W^{\mu\nu} W_{\mu\nu} + \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R \tag{2.1}$$

B is the hypercharge abelian gauge field; W is the gauge field of the local group $SU(2)$ of the isospin symmetry, which couples only to left-handed fields:

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{2.2a}$$

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig\left[W_\mu, W_\nu\right] \tag{2.2b}$$

$$W_\mu = W_\mu^a \tfrac{1}{2}\tau^a \tag{2.2c}$$

$$\left[\tfrac{1}{2}\tau^a, \tfrac{1}{2}\tau^b\right] = i\epsilon^{abc}\tfrac{1}{2}\tau^c \tag{2.2d}$$

$$D_\mu \psi_L = \partial_\mu \psi_L - igW_\mu \psi_L - ig' \frac{Y_L}{2} B_\mu \psi_L \tag{2.2e}$$

$$D_\mu \psi_R = \partial_\mu \psi_R - ig' \frac{Y_R}{2} B_\mu \psi_R \tag{2.2f}$$

where $Y_{L,R}$ is the hypercharge of the Dirac left-handed (resp. right-handed) spinors. In this way, the left-handed spinor belongs to the fundamental representation of $SU(2)_L$, whereas the right-handed one has no action under $SU(2)_L$; we can therefore arrange our lepton fields in a left-handed doublet and a right-handed singlet:

$$\psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}_{Y=-1} , \qquad \psi_R = (e_R)_{Y=-2} \tag{2.3}$$

the hypercharge being assigned by the Gell-Mann–Nishima formula: $Q = \frac{1}{2}\left(\tau^3 + Y\right)$.

Actually, this mass lagrangian is gauge invariant, whereas if we add the masses explicitely it is no longer so. Glashow tried the same to break explicitely gauge symmetry, adding

$$\mathcal{L}_{\text{mass}} = \frac{1}{2}\left[m_W^2 W_\mu^a W^{a\mu} + m_B^2 B_\mu B^\mu + 2m_{WB} W_\mu^3 B^\mu\right] + m\bar{\psi}\psi \tag{2.4}$$

We can separate the charged $W^\pm$ bosons from the neutral W^3 and B bosons by substituting:

$$W_\mu^\pm = \frac{W_\mu^1 \mp iW_\mu^2}{\sqrt{2}} \tag{2.5a}$$

$$\begin{aligned}\mathcal{L}_{\text{gauge}} = & - W_{\mu\nu}^\dagger W^{\mu\nu} + m_W^2 W_\mu^\dagger W^\mu - \frac{1}{4}\left[W_{\mu\nu}^3 W^{3\mu\nu} + B_{\mu\nu}B^{\mu\nu}\right] \\ & + \frac{1}{2}\left[m_W^2 W_\mu^3 W^{3\mu} + m_B^2 B_\mu B^\mu + 2m_{WB} W_\mu^3 B^\mu\right]\end{aligned} \tag{2.5b}$$

Eventually, we can define a mixing angle θ_W between W^3 and B and impose to an eigenstate to be massless, in order to recover the photon A:

$$Z_\mu = W_\mu^3 \cos\theta_W - B_\mu \sin\theta_W \tag{2.6a}$$

$$A_\mu = W_\mu^3 \sin\theta_W + B_\mu \cos\theta_W \tag{2.6b}$$

$$\begin{aligned}\mathcal{L}_{\text{gauge}} = & - W_{\mu\nu}^\dagger W^{\mu\nu} + m_W^2 W_\mu^\dagger W^\mu - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} \\ & + \frac{1}{2} m_Z^2 Z_\mu Z^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\end{aligned} \tag{2.6c}$$

where the τ^a are the Pauli matrices, $m_Z = m_W/\cos\theta_W$ and $F_{\mu\nu}$ is the ordinary electromagnetic strength tensor. For the interactions, we have

$$\mathcal{L}_{\text{int}} = g\left[\frac{1}{\sqrt{2}}W^+\mu J_+^\mu + W^-\mu J_-^\mu + \frac{1}{2\cos\theta_W}Z^+\mu J_Z^\mu\right] + eA_\mu J_{EM}^\mu \tag{2.7a}$$

$$J_+^\mu = \bar{\nu}_L\gamma^\mu e_L, \qquad J_-^\mu = (J_+^\mu)^\dagger = \bar{e}_L\gamma^\mu\nu_L \tag{2.7b}$$

$$J_{EM}^\mu = -\bar{e}\gamma^\mu e \tag{2.7c}$$

$$J_Z^\mu = \bar{\nu}_L\gamma^\mu\nu_L + \left(-1 + 2\sin^2\theta_W\right)\bar{e}_L\gamma^\mu e_L + 2\sin^2\theta_W\bar{e}_R\gamma^\mu e_R \tag{2.7d}$$

where the coupling g' can be removed by imposing the correct electromagnetic coupling: $g\sin\theta_W = g'\cos\theta_W = e$; the same happens for the μ and τ families. It is important to notice that, in the low-energy limit, the effective coupling for both charged and neutral currents is just the same:

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{g^2}{8\cos^2\theta_W m_Z^2} \tag{2.8}$$

At the time, Glashow theory had many unpleasant aspects. Phenomenologically, it predicted the existence of neutral currents, which would experimentally be discovered only in 1973; then, it was not clear how to extend the model to hadrons. Finally, from a theoretical point of view, the introduction of explicit (and large) mass terms spoils gauge invariance and make the whole theory non-renormalizable.

2.1 Higgs mechanism

In 1967, Weinberg and Salam [15, 16] solved this issue, introducing a doublet of scalar fields which allowed gauge bosons to receive a mass, without breaking gauge invariance. They added a scalar field whose auto-interactions induce a spontaneous symmetry breaking of the gauge group, thus providing effective mass terms to the gauge bosons, with a mechanism discovered by Higgs; Englert and Brout; Guralnik, Hagen and Kibble [17–19].

Thus, we add to the massless lagrangian (2.1) a scalar term in the fundamental representation of the gauge group:

$$\begin{aligned}\mathcal{L}_{\text{scalar}} &= \left(D_\mu\phi\right)^\dagger D^\mu\phi - V\left(\phi\right)\\ &= \left(\partial_\mu\phi^\dagger + ig\phi^\dagger W_\mu + ig'\frac{Y}{2}\phi^\dagger B_\mu\right)\left(\partial^\mu\phi - igW^\mu\phi - ig'\frac{Y}{2}B^\mu\phi\right) - V\left(\phi\right)\end{aligned} \tag{2.9}$$

$V\left(\phi\right)$ must be a potential with non-trivial minima, for example the mexican hat $V\left(\phi\right) = -\mu^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2$. Now, we choose a non-vanishing vacuum expectation value for the ϕ doublet and shift with respect to this value, namely

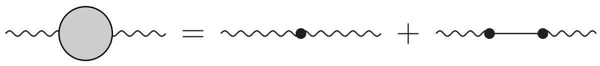
$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} -i\left(\phi^1 + i\phi^2\right) \\ v + h + i\phi^3 \end{pmatrix}_{Y=1}, \qquad \langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}_{Y=1} \tag{2.10}$$

The ϕ^1, ϕ^2 and ϕ^3 are the Goldstone bosons generated by the spontaneous breaking of the symmetry; h is the massive Higgs boson. The VEV is modified by the action of the gauge group, but is unchanged under the action of $Q = \frac{1}{2}\left(\tau^3 + Y\right)$. If we insert the (2.10) into (2.9), and use the basis of the mass eigenstates:

$$\begin{aligned}\mathcal{L}_{\text{scalar}} =& \partial_\mu\phi^\dagger\partial^\mu\phi + g^2 W^i_\mu W^{j\mu}\phi^\dagger\frac{\tau^i}{2}\frac{\tau^j}{2}\phi + g'^2 B_\mu B^\mu\phi^\dagger\frac{1}{2}\frac{1}{2}\phi\\ &+ 2gg'B_\mu W^{i\mu}\phi^\dagger\frac{\tau^i}{2}\frac{1}{2}\phi - 2igW^{i\mu}\partial_\mu\phi^\dagger\frac{\tau^i}{2}\phi - 2ig'B^\mu\partial_\mu\phi^\dagger\frac{1}{2}\phi\\ =& g^2 W^i_\mu W^{i\mu}\frac{v^2}{8} + g'^2 B_\mu B^\mu\frac{v^2}{8} - 2gg'B_\mu W^{3\mu}\frac{v^2}{4}\\ &+ 2gW^{i\mu}\partial_\mu\phi^\dagger\frac{\tau^i}{2}\langle\phi\rangle + 2g'B^\mu\partial_\mu\phi^\dagger\frac{1}{2}\langle\phi\rangle + \ldots\\ =& m_W^2 W^+_\mu W^{-\mu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu\\ &+ \frac{m_W}{2}\left(gW^-_\mu\partial^\mu\phi^+ + gW^+_\mu\partial^\mu\phi^-\right) + \frac{m_Z}{2}Z_\mu\partial^\mu\phi^3 + \ldots\end{aligned} \tag{2.11}$$

where $\phi^\pm = \left(\phi^1 \pm i\phi^2\right)/\sqrt{2}$, $m_W = \frac{1}{2}gv$, $m_Z = \frac{1}{2}v\sqrt{g^2 + g'^2}$; we recover the θ_W of (2.6) by imposing $\sin\theta_W = g'/\sqrt{g^2 + g'^2}$. In the dots we have hidden the gauge-Goldstone and the gauge-Higgs interaction terms.

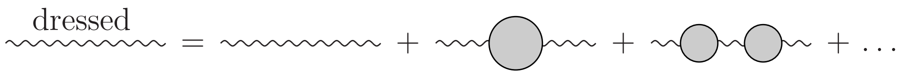
Thus, a spontaneous symmetry breaking can provide effective mass terms to the gauge bosons; moreover, compared to the Glashow model, we have a direct coupling between gauge bosons and Goldstone bosons (last row in (2.11)) which allow gauge invariance to be mantained. For example, we can calculate the dressed propagator of the Z boson in the Landau gauge, by inserting the mass term and a propagation of the massless ϕ^3 Goldstone boson:



$$= im_Z^2 g^{\mu\nu} + (m_Z k^\mu)\,\frac{i}{k^2}\,(-m_Z k^\mu)$$

$$= im_Z^2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) = im_Z^2 P_T^{\mu\nu}$$

P_T is a projector onto non-longitudinal polarization states; indeed $P_{T\mu\alpha}P_T^{\alpha\nu} = P_{T\mu}^{\ \ \nu}$.



$$= \frac{-i}{k^2}P_T^{\mu\nu} + \frac{-i}{k^2}P_T^{\mu\alpha} im_Z^2 P_{T\alpha\beta}\frac{-i}{k^2}P_T^{\beta\nu} + \dots$$

$$= \frac{-i}{k^2}P_T^{\mu\nu}\sum_n\left(\frac{m_Z^2}{k^2}\right)^n = \frac{-i}{k^2 - m_Z^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$$

Thus, in some way the Goldstone boson has been "eaten" by the Z, and a good transverse result is recovered; this is a sign that the Goldstone bosons are unphysical particles, whose action can be absorbed in massive gauge bosons. Actually, we will see how the Goldstone bosons can be removed from the theory. Besides, we have not yet understood how the symmetry breaking modifies the procedure of gauge fixing.

2.2 R_ξ gauges

We would like to choose a gauge so that the unpleasant direct coupling between Goldstone and gauge bosons can be absorbed. We start from the lagrangian of a general non-abelian gauge theory, coupled to a n-plet of real scalar fields in a representation of the gauge group.

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}^a\right)^2 + \left(D_\mu\phi\right)^2 - V\left(\phi\right) \tag{2.12a}$$

$$D_\mu\phi_i = \partial_\mu\phi_i + gA_\mu^a T_{ij}^a\phi_j\langle\phi\rangle = \phi_0 \tag{2.12b}$$

The T^a are in a skew-hermitian representation of the algebra. If the group is not simple (like for $SU(2)\otimes U(1)$), the gs need not be the same for every T^a. The

ϕ_0 contains the information about the pattern of symmetry breaking: the field fluctuations along the directions of the vectors $F_i^a = T_{ij}^a \phi_{0j}$ correspond to the Goldstone bosons; the orthogonal fluctuations correspond to the massive Higgs bosons. If we shift $\phi = \phi_0 + \phi'$, we find, for the quadratic terms in the lagrangian,

$$\begin{aligned}\mathcal{L} =& -\frac{1}{2} A_\mu^a \left(-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu\right) A_\nu^a + \frac{1}{2}\left(\partial_\mu\phi'\right)^2 \\ &+ g\partial^\mu \phi_i' A_\mu^a F_i^a + \frac{1}{2} g^2 F_i^a F_i^b A_\mu^a A^{b\mu} - \frac{1}{2} M_{ij}\phi_i'\phi_j'\end{aligned} \tag{2.13}$$

where $M_{ij} = \frac{\partial^2}{\partial\phi_i\partial\phi_j} V(\phi)\Big|_{\phi_0}$. Some remarks: the direct coupling between A and ϕ does not involve the massive Higgs boson (if h is the direction of the Higgs field, $F_h^a \phi_h' = 0$); the matrix M_{ij} is zero in the subspace of the Goldstone bosons (for the Goldstone theorem), whereas it provides a mass term to the Higgs boson. We consider now the functional integral and use the Faddeev-Popov gauge-fixing procedure:

$$\begin{aligned}Z =& \int \mathcal{D}A\mathcal{D}\phi' \exp\left[i\int \mathcal{L}\left(A,\phi'\right)\right] \\ =& C\int \mathcal{D}A\mathcal{D}\phi' \exp\left[i\int \mathcal{L}\left(A,\phi'\right) - \frac{1}{2}G^2\right] \mathrm{Det}\left(\frac{\delta G}{\delta\alpha}\right)\end{aligned} \tag{2.14}$$

where G is the gauge-fixing functional, α is the parameter of an infinitesimal gauge transformation, and C is a constant which contains the infinite volume of the group and can be omitted. We choose an *ad hoc* gauge-fixing functional to cancel the direct coupling $A\phi$:

$$G^a = \frac{1}{\sqrt{\xi}}\left(\partial^\mu A_\mu^a - \xi g F_i^a \phi_i'\right) \tag{2.15}$$

This choice for the gauge fixing for a spontaneously broken gauge theory is called R_ξ gauges (or renormalizable gauges). In particular, the gauge with $\xi = 1$ is the ‘t Hooft-Feynman gauge.

The quadratic part of the gauge-fixed lagrangian becomes:

$$\begin{aligned}\mathcal{L} =& -\frac{1}{2} A_\mu^a \left[\left(-g^{\mu\nu}\partial^2 + \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)\delta^{ab} - g^2 F_i^a F_i^b\right] A_\nu^b \\ &+ \frac{1}{2}\left(\partial_\mu\phi'\right)^2 - \frac{1}{2} M_{ij}\phi_i'\phi_j' - \frac{1}{2}\xi g^2 F_i^a F_j^a \phi_i'\phi_j'\end{aligned} \tag{2.16}$$

we can identify the mass terms:

$$g^2 F_i^a F_i^b = \left(m_A^2\right)^{ab} \tag{2.17a}$$

$$\xi g^2 F_i^a F_j^a = \xi \left(m_\phi^2\right)_{ij} \tag{2.17b}$$

We see that the Goldstone bosons have acquired a gauge-dependent mass; this is again a sign of their unphysical nature. We still have to construct the ghost lagrangian. The infinitesimal gauge transformation act on the fields so that:

$$\delta\phi_i = -\alpha^a(x)\, T_{ij}^a \phi_j \tag{2.18a}$$

$$\delta A_\mu^a = \frac{1}{g} D_\mu^{ab} \alpha^b(x) \tag{2.18b}$$

hence

$$\frac{\delta G^a}{\delta \alpha^b} = \frac{1}{\sqrt{\xi}} \left(\frac{1}{g}\partial^\mu D_\mu^{ab} + \xi g F_i^a T_{ij}^b \left(\phi_{0j} + \phi_j'\right)\right) \tag{2.19a}$$

$$\mathrm{Det}\, \frac{\delta G^a}{\delta \alpha^b} = \bar{\eta}^a \left(-\partial^\mu D_\mu^{ab} - \xi g^2 F_i^a F_i^b - \xi g^2 F_i^a T_i^b \phi_j'\right) \eta^b \tag{2.19b}$$

The ghosts have acquired the same mass term as the gauge bosons, times ξ. Moreover, with respect to the gauge-ghost vertex of the unbroken theory, we have added a coupling term between the ghosts and the scalar fields.

Finally, we can apply all this mechanism to the GWS theory. We can arrange the four scalar fields as in (2.10). The skew-hermitian T^as are

$$T^a = \begin{cases} -i\dfrac{\tau^a}{2} & \text{if } a = 1,2,3 \\ -i\dfrac{1}{2} & \text{if } a = 4 \text{ (hypercharge)} \end{cases} \tag{2.20}$$

and

$$gF_i^a = \frac{v}{2} \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \\ 0 & 0 & -g' \end{pmatrix} \tag{2.21}$$

The index $i = 4$ would correspond to the Higgs field, and we would have $gF_4^a = 0$; so the index i can be considered to run only on $1, 2, 3$.

Thus, the gauge boson mass matrix is

$$\left(m_A^2\right)^{ab} = g^2 F_i^a F_i^b = \frac{v^2}{4}\begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix} \tag{2.22}$$

which can be diagonalized with the transformations in (2.6), provided that

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \tag{2.23}$$

The eigenvalues are $m_W^2 = \frac{1}{4}g^2v^2$, $m_Z^2 = \frac{1}{4}\left(g^2+g'^2\right)v^2$, and $m_A^2 = 0$.

Hence, in the basis of mass eigenstates, the gauge bosons decorrelate (meaning that there is no mixed 2-vertex), and the propagators are

$$i\Delta^{\mu\nu} = \frac{1}{q^2-m^2}\left(g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2-\xi m^2}\right) \tag{2.24}$$

where $m = m_W, m_Z, 0$ for the W and Z bosons, and for the photon. It is important to notice that the propagator is $O\left(q^{-2}\right)$ for whatever finite value of ξ, hence the diagrams in R_ξ gauge have a good ultraviolet behaviour, which allowed 't Hooft to prove the renormalizability of spontaneously broken gauge theories [20, 21].

In the matter of Goldstone bosons, the mass matrix is

$$\xi\left(m_\phi^2\right)_{ij} = \xi g^2 F_i^a F_j^a = \xi\frac{v^2}{4}\begin{pmatrix} g^2 & 0 & 0 \\ 0 & g^2 & 0 \\ 0 & 0 & g^2+g'^2 \end{pmatrix} \tag{2.25}$$

and the propagators are

$$\Delta = \frac{i}{q^2-\xi m^2} \tag{2.26}$$

with $m = m_W$ for $\phi^\pm$, and $m = m_Z$ for ϕ^3. On the contrary, the physical Higgs field propagates independently with a mass determined only by the potential $V(x)$ (and no ξ dependence).

Finally, the ghost propagators are

$$\Delta = \frac{i}{q^2 - \xi m^2} \tag{2.27}$$

where $m = m_W, m_Z, 0$ for $\eta^{\pm}$, η^Z, and η^{γ} [1].

2.3 Unitary gauge

It is now clear that the Goldstone bosons are unphysical particles: their mass depends on a ficticious parameter like ξ. Then, if we take the limit $\xi \to \infty$, we expect the Goldstone bosons and the Faddeev-Popov ghosts to take an infinite mass, and so to decouple from the theory. Actually, this is true, and the correspondent gauge is called unitary gauge.

We can simply choose a gauge-fixing functional to impose on the theory the vanishing of the Goldstone bosons:

$$G^a = \begin{cases} \phi'_a & \text{for } a = 1,2,3 \\ \dfrac{1}{\sqrt{\xi}}\partial^\mu A_\mu & \text{for } a = 4 \end{cases} \tag{2.28}$$

(the fourth condition being the usual Lorentz gauge fixing for the massless photon).

For the quadratic part of the lagrangian in the basis of mass eigenstates, we can suppress all the Goldstone bosons contributions, leaving only the physical Higgs field:

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}Z_\mu\left(-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu - m_Z^2 g^{\mu\nu}\right)Z_\nu - W_\mu^\dagger\left(-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu - m_W^2 g^{\mu\nu}\right)W_\nu \\ &-\frac{1}{2}A_\mu\left(-g^{\mu\nu}\partial^2 + \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)A_\nu + \frac{1}{2}\left(\partial_\mu h\right)^2 - \frac{1}{2}m_h h^2 \end{aligned} \tag{2.29}$$

where $m_W = \frac{1}{2}gv$, $m_Z = \frac{1}{2}v\sqrt{g^2 + g'^2}$, and $m_h = \frac{\partial^2}{\partial h^2}V(\phi)\Big|_{\phi_0}$.

[1] One could expect that η^γ would decouple from the theory as in an abelian unbroken theory. However, the presence of the coupling between the scalar fields and the ghost in the gauge-fixing functional does not allow the abelian ghost to decouple; even more so in a non-abelian theory with an abelian subgroup like $SU(2) \otimes U(1)$.

The propagators of mass eigenstates are indeed

$$
\begin{aligned}
i\Delta^{\mu\nu} =& \frac{1}{q^2 - m^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m^2}\right) && \text{for the massive bosons} \\
i\Delta^{\mu\nu} =& \frac{1}{q^2 - m^2}\left(g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}\right) && \text{for the photon}
\end{aligned}
\tag{2.30}
$$

where $m = m_W, m_Z$. The photon propagator has the usual form of the unbroken theory, whereas the other propagators have the correct form for a spin-1 massive particle; in the rest frame indeed, the numerator becomes

$$
\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m^2}\right)_{\text{rest}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\tag{2.31}
$$

which is the correct form for the sum over the 3 physical polarizations (ϵ_+, ϵ_- and ϵ_L). Besides, we would obtain the same propagator by taking the limit $\xi \to \infty$ of (2.24), as predicted at the beginning of this section.

In the matter of ghosts, the infinitesimal gauge transformation can be expressed in the basis of mass eigenstates [22]:

$$
\delta\phi^\pm = \alpha^\pm \frac{g}{2}(v+h) + O(\phi) \tag{2.32a}
$$

$$
\delta\phi^3 = \alpha^3 \frac{g}{2\cos\theta_W}(v+h) + O(\phi) \tag{2.32b}
$$

$$
\delta\left(\partial^\mu A_\mu\right) = -\partial^2\alpha^\gamma + O(\phi) + O\left(W^\pm\right) \tag{2.32c}
$$

We isolated the terms $O(\phi)$ because, after the functional derivative, they are forced to zero by $\delta\left(G^a\right)$. Moreover, the ghost of the photon field η^γ is still coupled to the $W^\pm$ bosons, but only with a vertex with an outgoing η^γ and an ingoing $\eta^\pm$; so it cannot run inside a loop, and it can be integrated out. Thus, the functional determinant can be expressed as

$$
\text{Det}\,\frac{\delta G^a(x)}{\delta\alpha^b(y)} = \text{Det}\left[\frac{g}{2}(v+h)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\cos\theta_W \end{pmatrix}\delta^4(x-y)\right]
\tag{2.33}
$$

The functional determinant does not contain any derivative, so the resulting ghost fields do not propagate, but lead only to local contributions. However, the determinant can be computed directly without adding any ghost field:

$$
\begin{aligned}
\operatorname{Det} \frac{\delta G^a(x)}{\delta \alpha^b(y)} &= \exp\left[\delta^4(0) \int dx \det \ln \frac{g}{2}(v+h) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\cos\theta_W \end{pmatrix}\right] \\
&= \exp\left[i\delta^4(0) \int dx (-3i) \ln\left(1 + \frac{h}{v}\right) + \text{const}\right]
\end{aligned}
\tag{2.34}
$$

That adds an effective term to the lagrangian:

$$
\mathcal{L}_{\text{FP}} = -3i\,\delta^4(0) \ln\left(1 + \frac{h}{v}\right) \tag{2.35}
$$

This term could seem quite obscure; however, it can be obtained in the limit $\xi \to \infty$; we write the contribution of a loop of ghosts with N external Higgs lines (Figure 2.1):

$$
\begin{aligned}
&-\int \frac{d^4p}{(2\pi)^4} \left(-\frac{i}{2}\xi g m_W\right)^N \prod_i^N \frac{i}{(p+\Delta p_i)^2 - \xi m_W^2} \\
&\xrightarrow{\xi\to\infty} -\int \frac{d^4p}{(2\pi)^4} \left(-\frac{g}{2m_W}\right)^N = -\delta^4(0) \left(-\frac{g}{2m_W}\right)^N
\end{aligned}
\tag{2.36}
$$

If we sum the contributions of the ghosts which can run in the loop (η^+, η^-, η^Z), and after some combinatorial considerations [2], we get to the term (2.35)

[2] We can connect the Higgs lines to the ghost loop in $(N-1)!$ ways; the loop can be replaced by an effective vertex which instead can be connected to Higgs lines in $N!$ ways. So we must add a factor $(N-1)!/N! = 1/N$ to (2.36); if we sum over N, we get to (2.35).

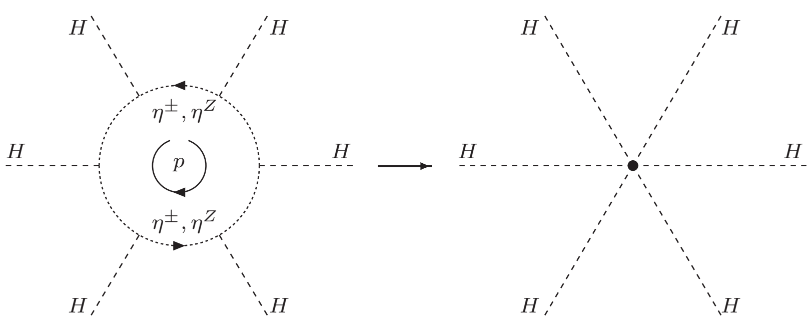


FIGURE 2.1: The extra-term (2.35) which arises from ghost loops in unitary gauge

Thus, in unitary gauge we can remove all the unphysical degrees of freedom, and we can deal only with the photon, the massive gauge bosons (with the 3 physical polarization states) and the Higgs boson. However, the loops with only Higgs bosons on the external legs need the adding of the divergent term (2.35). We do not need this term in the calculation of $H \to \gamma\gamma$.

We end this section with a consideration about power counting: unlike the R_ξ gauges, the propagators of the gauge bosons are $O(q^0)$ instead of $O(q^{-2})$. Therefore, in unitary gauge loop integrals have higher divergent terms than R_ξ gauges; however, gauge invariance should grant a cancellation of the extra-divergent terms. We will provide some explicit examples in the following chapters.

2.4 $H \to \gamma\gamma$

Before tackling the calculations by Gastmans *et al.*, we mean to show the final expressions of the amplitude:

$$\mathcal{M} = \frac{e^2 g}{(4\pi)^2 m_W} \left[(k_1 \cdot k_2) g_{\mu\nu} - k_{1\nu} k_{2\mu}\right] \epsilon_1^\mu \epsilon_2^\nu \left(F_W + \sum_f N_C \, e_f^2 F_f \right) \qquad (2.37)$$

where e_f is the charge carried by the fermions which run in the loop, N_C is the

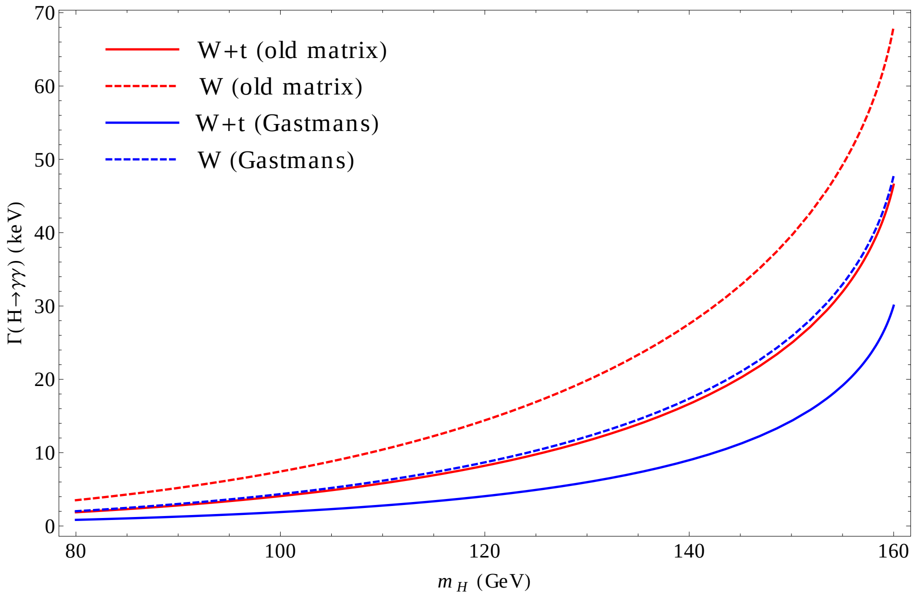


FIGURE 2.2: Plot of $\Gamma\,(H \to \gamma\gamma)$ as a function of m_H. The Gastmans *et al.*'s amplitude halves the contribution of the W loop.

number of colors (3 for the quarks, 1 for the leptons), and F_W, F_f are

$$(F_W)_{\text{old}} = 2 + 3\tau_W^{-1} + 3\tau_W^{-1}\left(2 - \tau_W^{-1}\right) f(\tau_W) \tag{2.38a}$$

$$(F_W)_{\text{new}} = 3\tau_W^{-1} + 3\tau_W^{-1}\left(2 - \tau_W^{-1}\right) f(\tau_W) \tag{2.38b}$$

$$F_f = -2\tau_f^{-1}\left[1 + \left(1 - \tau_f^{-1}\right) f(\tau_f)\right] \tag{2.38c}$$

where $\tau_i = \frac{m_H^2}{4m_i^2}$, $i = f, W$ and

$$f(\tau_i) = \begin{cases} \arcsin^2(\sqrt{\tau_i}) & \text{for} \quad \tau_i \leq 1 \\ -\frac{1}{4}\left[\ln \frac{1+\sqrt{1-\tau_i^{-1}}}{1-\sqrt{1-\tau_i^{-1}}} - i\pi\right]^2 & \text{for} \quad \tau_i > 1 \end{cases} \tag{2.39}$$

As required, the matrix elements acquire an imaginary part via the $f(\tau_i)$ when $\tau_i > 1$, i.e. when m_H is above the production threshold for two fermions/W. For the partial width, we obtain

$$\Gamma(H \to \gamma\gamma) = \frac{G\alpha^2 m_H^3}{128\sqrt{2}\pi^3} |F|^2 \tag{2.40}$$

with $F = F_W + \sum_f N_C e_f^2 F_f$.

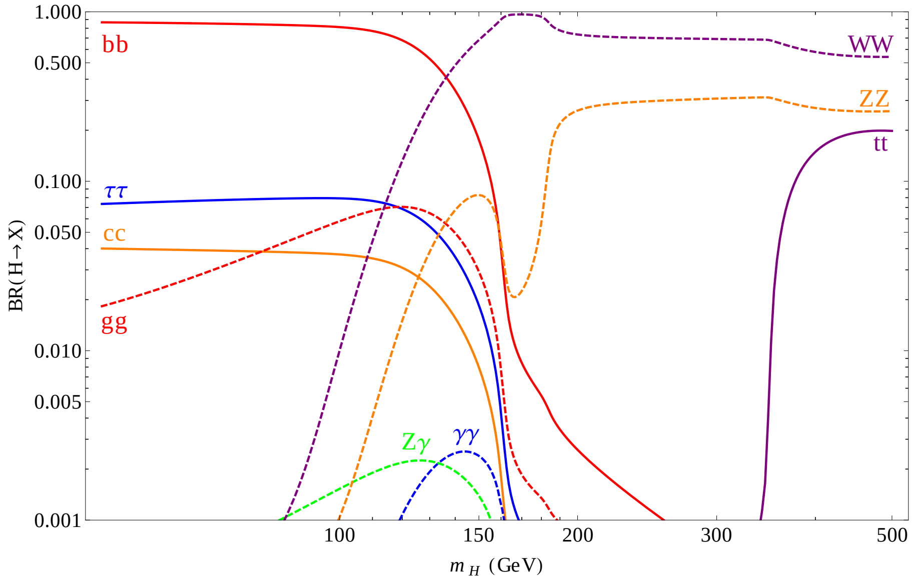


FIGURE 2.3: The branching ratios for the SM Higgs boson as a function of m_H. Although the $H \to \gamma\gamma$ channel has a $\mathcal{BR} \approx 10^{-3}$, the signal has a background smaller than other channels.

The radiative corrections to this partial width only affect the top quark loop and neither the W loop nor the final state photons. The QCD corrections are below 3% [23] and thus can be neglected. In Figure 2.2 we plot the partial widths as a function of m_H, both via the W loop alone and together with the top loop (which gives by far the biggest contribution among the fermions). Besides, for F_W we use both the old expression (2.38a) and the new one (2.38b).

Among the Higgs decay channels, the $H \to \gamma\gamma$ can be considered quite rare, with a branching ratio of $\sim 10^{-3}$ (Figure 2.3). Nonetheless, it is reckoned the most promising channel for the search of a Higgs boson below 150 GeV. Indeed, the diphoton mass resolution is very good, between 1 and 2%; the signature in this channel is two high E_{T} isolated photons (with two additional high p_{T} jets if the Higgs has been produced via Vector Boson Fusion); the background is dominated by the irreducible two photon QCD production, with also a relevant contribution from events in which at least one of the two identified photons is a jet faking a photon. In Figure 2.4 we show a plot by CMS [24] to give an idea of the ratio between the Higgs signal and the QCD background.

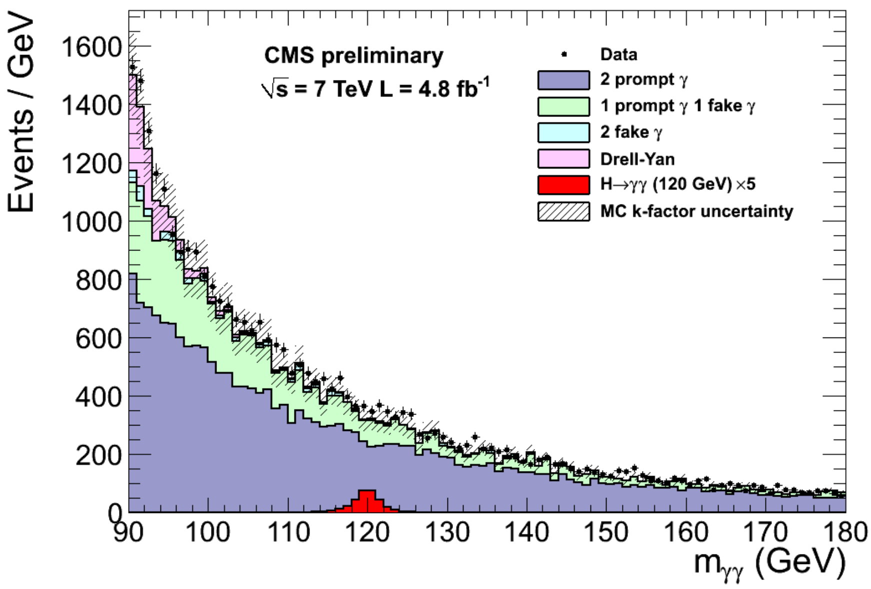


FIGURE 2.4: Di-photon mass spectrum from CMS preliminars. Data are shown together with the background MC prediction. The expected Higgs signal at 120 GeV is also shown superimposed and scaled by a factor 5.

The new result by Gastmans *et al.* suppresses the W loop, so that the partial width into photons is halved:

$$\Gamma(H \to \gamma\gamma)_{\text{Gastmans}} \approx 0.48 \times \Gamma(H \to \gamma\gamma)_{\text{old}} \tag{2.41}$$

Hence, if the new result were correct, the signal would be even twice smaller, challenging the experimental search.

Chapter 3

The body evidence

We now sum up the main points of Gastmans *et al.*'s calculation. First of all, they cast some doubts on the reliability of the old results of Refs. [3, 4]:

- The calculations were performed in the limit $m_H << m_W$, while nowadays we expect $m_H \approx 1.5\, m_W$.
- The use of the 't Hooft-Feynman gauge introduces a lot of unphysical particles (would-be Goldstones and ghosts).
- Dimensional regularization was used in Ref. [3]

Since the photon is massless, there is no direct coupling of the Higgs boson to the photon in the lagrangian of the Standard Model, and therefore the renormalizability of the standard model implies that the one-loop amplitude must be finite (there would be no counterterm $H\gamma\gamma$ to absorb one-loop divergences). For this reason, Gastmans *et al.* decided to repeat the calculations without using any regulator. Moreover, they let m_H arbitrary and choose unitary gauge to avoid any possible ambiguity with the use of unphysical particles.

In the unitary gauge we have only three Feynman diagrams we report in Figure 3.1. The integrals we expect will badly diverge by naive power counting (see Sec. 2.3); hence, we must be particularly careful to choose the loop momentum in all diagrams, in order to avoid shifting it and obtaining undesired surface terms. This has been discussed by Gastmans *et al.* in Ref. [1], on the basis of some examples in QED [25], and $\lambda\phi^4$ [26]. In the following, k_1 and k_2 are the on-shell momenta of the outgoing photons, ϵ_1 and ϵ_2 their polarization vectors. We have

$$k_1^2 = k_2^2 = 0 \tag{3.1a}$$
$$k_1 \cdot k_2 = \tfrac{1}{2}\left(k_1 + k_2\right)^2 = \tfrac{1}{2} m_H^2 \tag{3.1b}$$
$$k_1 \cdot \epsilon_1 = k_2 \cdot \epsilon_2 = 0 \tag{3.1c}$$

The derivation of the matrix elements is now straightforward:

$$i\mathcal{M} = \left(i\mathcal{M}_1^{\mu\nu} + i\mathcal{M}_2^{\mu\nu} + i\mathcal{M}_3^{\mu\nu}\right)\epsilon_{1\mu}\epsilon_{2\nu} \tag{3.2}$$

with:

$$\begin{aligned}
i\mathcal{M}_1^{\mu\nu} =& \frac{e^2 g\, m_W}{(2\pi)^4} \int d^4k\, \frac{g_\alpha^\beta - \left(k + \frac{1}{2}k_1 + \frac{1}{2}k_2\right)_\alpha \left(k + \frac{1}{2}k_1 + \frac{1}{2}k_2\right)^\beta / m_W^2}{\left(k + \frac{1}{2}k_1 + \frac{1}{2}k_2\right)^2 - M^2 + i\epsilon} \\
&\times \frac{g^{\rho\sigma} - \left(k - \frac{1}{2}k_1 + \frac{1}{2}k_2\right)^\rho \left(k - \frac{1}{2}k_1 + \frac{1}{2}k_2\right)^\sigma / m_W^2}{\left(k - \frac{1}{2}k_1 + \frac{1}{2}k_2\right)^2 - m_W^2 + i\epsilon} \\
&\times \frac{g^{\alpha\gamma} - \left(k - \frac{1}{2}k_1 - \frac{1}{2}k_2\right)^\alpha \left(k - \frac{1}{2}k_1 - \frac{1}{2}k_2\right)^\gamma / m_W^2}{\left(k - \frac{1}{2}k_1 - \frac{1}{2}k_2\right)^2 - m_W^2 + i\epsilon} \\
&\times \left[\left(k + \tfrac{3}{2}k_1 + \tfrac{1}{2}k_2\right)_\rho g_{\beta\mu} + \left(k - \tfrac{3}{2}k_1 + \tfrac{1}{2}k_2\right)_\beta g_{\mu\rho} + \left(-2k - k_2\right)_\mu g_{\rho\beta}\right] \\
&\times \left[\left(k - \tfrac{1}{2}k_1 - \tfrac{3}{2}k_2\right)_\sigma g_{\gamma\nu} + \left(k - \tfrac{1}{2}k_1 + \tfrac{3}{2}k_2\right)_\gamma g_{\nu\sigma} + \left(-2k + k_1\right)_\nu g_{\sigma\gamma}\right]
\end{aligned} \tag{3.3}$$

$$i\mathcal{M}_2^{\mu\nu} = i\mathcal{M}_1^{\mu\nu}\left(\mu \leftrightarrow \nu,\, k_1 \leftrightarrow k_2\right) \tag{3.4}$$

$$\begin{aligned}
i\mathcal{M}_3^{\mu\nu} =& \frac{e^2 g\, m_W}{(2\pi)^4} \int d^4k\, \frac{g_\alpha^\beta - \left(k + \frac{1}{2}k_1 + \frac{1}{2}k_2\right)_\alpha \left(k + \frac{1}{2}k_1 + \frac{1}{2}k_2\right)^\beta / m_W^2}{\left(k + \frac{1}{2}k_1 + \frac{1}{2}k_2\right)^2 - M^2 + i\epsilon} \\
&\times \frac{g^{\alpha\gamma} - \left(k - \frac{1}{2}k_1 - \frac{1}{2}k_2\right)^\alpha \left(k - \frac{1}{2}k_1 - \frac{1}{2}k_2\right)^\gamma / m_W^2}{\left(k - \frac{1}{2}k_1 - \frac{1}{2}k_2\right)^2 - m_W^2 + i\epsilon} \\
&\times \left[2\, g_{\mu\nu}\, g_{\beta\gamma} - g_{\mu\beta}\, g_{\nu\gamma} - g_{\mu\gamma}\, g_{\nu\beta}\right]
\end{aligned} \tag{3.5}$$

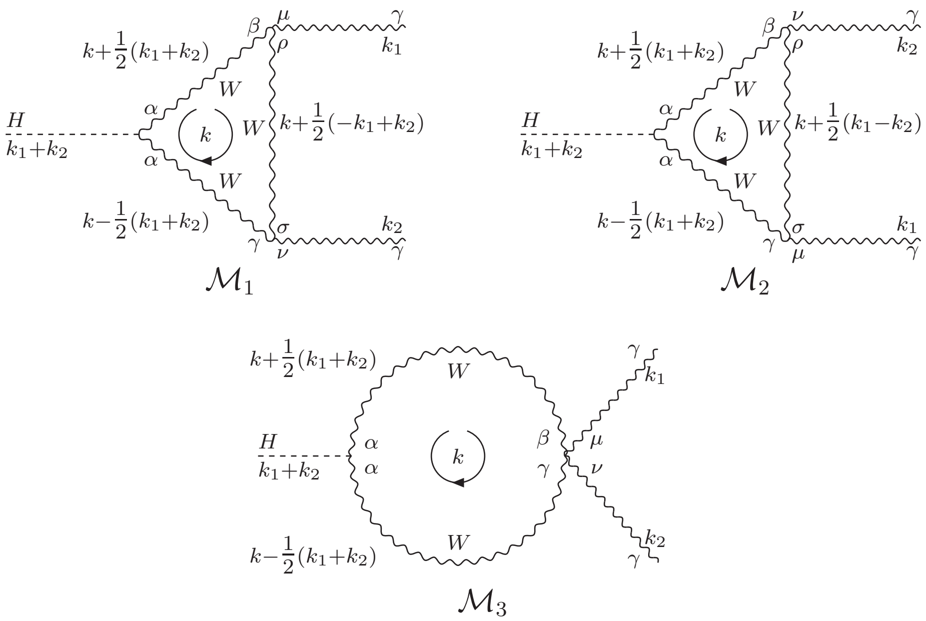


FIGURE 3.1: The one-loop diagrams with virtual Ws in the unitary gauge that contribute to the amplitude for $H \to \gamma\gamma$.

3.1 The evaluation

We introduce some notations to simplify the calculations of the matrix elements. We call p_1, p_2 and p_3 the momenta of the W in $\mathcal{M}_1$, so that

$$\begin{array}{lll} p_1 = k + \frac{1}{2}\left(k_1 + k_2\right), & p_2 = k + \frac{1}{2}\left(-k_1 + k_2\right), & p_3 = k - \frac{1}{2}\left(k_1 + k_2\right) \\ D_1 = p_1^2 - m_W^2 + i\epsilon, & D_2 = p_2^2 - m_W^2 + i\epsilon, & D_3 = p_3^2 - m_W^2 + i\epsilon \end{array} \tag{3.6}$$

Then we call $V_{\alpha\beta\gamma}$ the vertex $WW\gamma$:

$$V_{\alpha\beta\gamma}\left(p, q, r\right) = \left(q - r\right)_\alpha g_{\beta\gamma} + \left(r - p\right)_\beta g_{\gamma\alpha} + \left(p - q\right)_\gamma g_{\alpha\beta} \tag{3.7}$$

with $p + q + r = 0$. If we put the momentum of one outgoing photon on the mass-shell, and consider $k_{1\mu}$ to vanish when contracted with $\epsilon_{1\mu}$, we obtain:

$$p_1^\alpha V_{\alpha\mu\gamma}\left(p_1, -k_1, p_2\right) = p_2^2 g_{\mu\gamma} - p_{2\mu} p_{2\gamma} \tag{3.8}$$

and

$$p_2^\gamma p_1^\alpha V_{\alpha\mu\gamma}\left(p_1, -k_1, p_2\right) = 0 \tag{3.9}$$

Now, the numerators of the matrix elements contain inverse powers of m_W, which derive from the propagators of the W in the unitary gauge; we can expand the products and order by m_W. Each factor of m_W^{-2} comes with a product of two loop momenta, so that a term with m_W^{-n} will have a superficial degree of divergence of n. Moreover, since by naive power counting in R_ξ gauges all the diagrams diverge logarithmically, we expect the divergences more than logarithmic to vanish also in unitary gauge. In the following, we will omit the constant factor $e^2 g\, m_W/(2\pi)^4$ and the polarization vectors.

Let us start from the highest divergent term m_W^{-6}, which appears in $\mathcal{M}_1$ and $\mathcal{M}_2$:

$$i\mathcal{M}_1^{(6)} = -\frac{1}{m_W^6}\int d^4k\, \frac{p_1^\alpha p_1^\beta V_{\beta\mu\rho}\left(p_1,-k_1,-p_2\right) p_2^\rho p_2^\sigma V_{\sigma\nu\gamma}\left(p_2,-k_2,-p_3\right) p_3^\gamma p_{3\alpha}}{D_1 D_2 D_3} = 0 \tag{3.10}$$

for the (3.9). Similarly for $\mathcal{M}_2^{(6)}$.

The m_W^{-4} terms can be obtained with the product of two longitudinal parts of the propagators; if we choose both longitudinal parts of the propagators linked to a $WW\gamma$ vertex, the integral will vanish for (3.9). So, we have only

$$\begin{aligned} i\mathcal{M}_1^{(4)} =& \frac{1}{m_W^4}\int d^4k\, \frac{p_1^\alpha p_1^\beta V_{\beta\mu\rho}\left(p_1,-k_1,-p_2\right) g^{\rho\sigma} V_{\sigma\nu\gamma}\left(p_2,-k_2,-p_3\right) p_3^\gamma p_{3\alpha}}{D_1 D_2 D_3} \\ =& \frac{1}{m_W^4}\int d^4k\, \frac{p_1^\alpha \left(p_2^2 g_{\mu\rho} - p_{2\mu} p_{2\rho}\right) g^{\rho\sigma} V_{\sigma\nu\gamma}\left(p_2,-k_2,-p_3\right) p_3^\gamma p_{3\alpha}}{D_1 D_2 D_3} \\ =& \frac{1}{m_W^4}\int d^4k\, \frac{\left(p_1\cdot p_3\right) p_2^2 V_{\mu\nu\gamma}\left(p_2,-k_2,-p_3\right) p_3^\gamma}{D_1 D_2 D_3} \\ =& \frac{1}{m_W^4}\int d^4k\, \frac{\left(p_1\cdot p_3\right) V_{\mu\nu\gamma}\left(p_2,-k_2,-p_3\right) p_3^\gamma}{D_1 D_3} \\ &+ \frac{1}{m_W^2}\int d^4k\, \frac{\left(p_1\cdot p_3\right) V_{\mu\nu\gamma}\left(p_2,-k_2,-p_3\right) p_3^\gamma}{D_1 D_2 D_3} \end{aligned} \tag{3.11}$$

We will analyze the latter integral later with the other m_W^{-2} terms. In terms of external momenta, we have

$$\begin{aligned} &\frac{1}{m_W^4}\int d^4k\, \frac{\left(p_1\cdot p_3\right) V_{\mu\nu\gamma}\left(p_2,-k_2,-p_3\right) p_3^\gamma}{D_1 D_3} \\ =& \frac{1}{m_W^4}\int d^4k\, \frac{\left(p_1\cdot p_3\right) \left(p_2^2 g_{\mu\nu} - p_{2\mu} p_{2\nu}\right)}{D_1 D_3} \\ =& \frac{1}{m_W^4}\int d^4k\, \frac{\left(k^2 - \frac{1}{2}k_1\cdot k_2\right)}{D_1 D_3} \\ &\times \left(\left(k^2 - \frac{1}{2}k_1\cdot k_2 - k\cdot k_1 + k\cdot k_2\right) g_{\mu\nu} - \left(k+\tfrac{1}{2}k_2\right)_\mu \left(k-\tfrac{1}{2}k_1\right)_\nu\right) \end{aligned} \tag{3.12}$$

Therefore we add the m_W^{-4} terms of the crossed diagram $\mathcal{M}_2$:

$$
\begin{aligned}
(i\mathcal{M}_1 + i\mathcal{M}_2)^{(4)} =& \frac{1}{m_W^4} \int d^4k \, \frac{\left(k^2 - \frac{1}{2} k_1 \cdot k_2\right)}{D_1 D_3} \\
& \times \left(\left(2k^2 - k_1 \cdot k_2\right) g_{\mu\nu} - 2k_\mu k_\nu + \tfrac{1}{2} k_{1\nu} k_{2\mu} \right)
\end{aligned}
\tag{3.13}
$$

The $\mathcal{M}_3$ diagram is still to be to considered: taking the longitudinal part of both propagators, we get

$$
\begin{aligned}
i\mathcal{M}_3^{(4)} =& \frac{1}{m_W^4} \int d^4k \, \frac{\left(k^2 - \frac{1}{2} k_1 \cdot k_2\right)}{D_1 D_3} \\
& \times \left(- \left(2k^2 - k_1 \cdot k_2\right) g_{\mu\nu} + 2k_\mu k_\nu - \tfrac{1}{2} k_{1\nu} k_{2\mu} \right)
\end{aligned}
\tag{3.14}
$$

so that the sum of all m_W^{-4} terms vanishes identically.

The analysis of the m_W^{-2} is rather trickier. In $\mathcal{M}_1$, we start by choosing the longitudinal part of the third propagator:

$$
\begin{aligned}
i\mathcal{M}_1^{(2),3} =& - \frac{1}{m_W^2} \int d^4k \, \frac{p_3^\gamma p_3^\beta V_{\beta\mu\rho}\left(p_1, -k_1, -p_2\right) V^\rho{}_{\nu\gamma}\left(p_2, -k_2, -p_3\right)}{D_1 D_2 D_3} \\
=& - \frac{1}{m_W^2} \int d^4k \, \frac{p_3^\beta V_{\beta\mu\rho}\left(p_1, -k_1, -p_2\right)\left(p_2^2 g_\nu^\rho p_2^\rho - p_{2\nu}\right)}{D_1 D_2 D_3} \\
=& - \frac{1}{m_W^2} \int d^4k \, \frac{p_3^\beta V_{\beta\mu\nu}\left(p_1, -k_1, -p_2\right)}{D_1 D_3} \\
& + \frac{1}{m_W^2} \int d^4k \, \frac{p_3^\beta V_{\beta\mu\rho}\left(p_1, -k_1, -p_2\right) p_2^\rho p_{2\nu}}{D_1 D_2 D_3} \\
& - \int d^4k \, \frac{p_3^\beta V_{\beta\mu\nu}\left(p_1, -k_1, -p_2\right)}{D_1 D_2 D_3} \\
=& A_1^3 + B_1^3 + C_1^3
\end{aligned}
\tag{3.15}
$$

We work on the first integral. We explicit the external momenta:

$$
\begin{aligned}
A_1^3 =& - \frac{1}{m_W^2} \int d^4k \, \frac{p_3^\beta V_{\beta\mu\nu}\left(p_1, -k_1, -p_2\right)}{D_1 D_3} \\
=& - \frac{1}{m_W^2} \int d^4k \, \frac{p_3^\beta}{D_1 D_3} \Big[\left(k - \tfrac{3}{2} k_1 + \tfrac{1}{2} k_2\right)_\beta g_{\mu\nu} \\
& + \left(-2k - k_2\right)_\mu g_{\nu\beta} + \left(k + \tfrac{3}{2} k_1 + \tfrac{1}{2} k_2\right)_\nu g_{\beta\mu} \Big]
\end{aligned}
\tag{3.16}
$$

This can be combined with the analgous term of $\mathcal{M}_2$ obtained with the substitution $\mu \leftrightarrow \nu, k_1 \leftrightarrow k_2$:

$$
\begin{aligned}
A_2^1 =& -\frac{1}{m_W^2}\int d^4k\,\frac{p_3^\beta V_{\beta\nu\mu}\left(p_1,-k_2,-p_2\right)}{D_1D_3} \\
=& -\frac{1}{m_W^2}\int d^4k\,\frac{p_3^\beta}{D_1D_3}\Big[\left(k+\tfrac{1}{2}k_1-\tfrac{3}{2}k_2\right)_\beta g_{\nu\mu} \\
&+\left(-2k-k_1\right)_\nu g_{\mu\beta}+\left(k+\tfrac{1}{2}k_1+\tfrac{3}{2}k_2\right)_\mu g_{\beta\nu}\Big]
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
A_1^3+A_2^1 =& -\frac{1}{m_W^2}\int d^4k\,\frac{p_3^\beta}{D_1D_3}\Big[\left(2k-k_1-k_2\right)_\beta g_{\nu\mu} \\
&+\left(-k+\tfrac{1}{2}k_1+\tfrac{1}{2}k_2\right)_\nu g_{\mu\beta}+\left(-k+\tfrac{1}{2}k_1+\tfrac{1}{2}k_2\right)_\mu g_{\beta\nu}\Big] \\
=& -\frac{1}{m_W^2}\int d^4k\,\frac{p_3^\beta p_3^\alpha\left(2g_{\alpha\beta}g_{\nu\mu}-g_{\alpha\nu}g_{\mu\beta}-g_{\alpha\mu}g_{\beta\nu}\right)}{D_1D_3}
\end{aligned}
\tag{3.18}
$$

Similarly, we can choose in $\mathcal{M}_1$ the longitudinal part of the first propagator, combine it with the correspondent term in $\mathcal{M}_2$, and get

$$
\begin{aligned}
A_2^3+A_1^1 =& -\frac{1}{m_W^2}\int d^4k\,\frac{p_1^\beta}{D_1D_3}\Big[\left(2k+k_1+k_2\right)_\beta g_{\nu\mu} \\
&+\left(-k-\tfrac{1}{2}k_1-\tfrac{1}{2}k_2\right)_\nu g_{\mu\beta}+\left(-k-\tfrac{1}{2}k_1-\tfrac{1}{2}k_2\right)_\mu g_{\beta\nu}\Big] \\
=& -\frac{1}{m_W^2}\int d^4k\,\frac{p_1^\beta p_1^\alpha\left(2g_{\alpha\beta}g_{\nu\mu}-g_{\alpha\nu}g_{\mu\beta}-g_{\alpha\mu}g_{\beta\nu}\right)}{D_1D_3}
\end{aligned}
\tag{3.19}
$$

We recover a term proportional to the 4-vertex $WW\gamma\gamma$, and indeed

$$
A_1^3+A_2^1+A_2^3+A_1^1+i\mathcal{M}_3^{(2)}=0 \tag{3.20}
$$

At the order m_W^{-2} for $\mathcal{M}_1$, we still have to choose the transverse polarization of the propagator which links the photons, the B terms (3.15) for the other two choices, and finally the term in (3.11); specularly for $\mathcal{M}_2$. We have:

$$\begin{aligned}
i\mathcal{M}_1^{(2)} =& \frac{1}{m_W^2}\int\frac{d^4k}{D_1D_2D_3}\Bigg[\left(p_1\cdot p_3\right)V_{\mu\nu\gamma}\left(p_2,-k_2,-p_3\right)p_3^{\gamma} \\
& - p_2^{\rho}p_2^{\sigma}V_{\beta\mu\rho}\left(p_1,-k_1,-p_2\right)g^{\beta\gamma}V_{\sigma\nu\gamma}\left(p_2,-k_2,-p_3\right) \\
& + p_3^{\beta}V_{\beta\mu\rho}\left(p_1,-k_1,-p_2\right)p_2^{\rho}p_2^{\nu} + p_1^{\gamma}V_{\sigma\nu\gamma}\left(p_2,-k_2,-p_3\right)p_2^{\sigma}p_2^{\mu}\Bigg] \\
=& \frac{1}{m_W^2}\int\frac{d^4k}{D_1D_2D_3}\Bigg[\left(p_1\cdot p_3\right)\left(p_2^2g_{\mu\nu}+p_{2\mu}p_{2\nu}\right) \\
& + p_1^2p_3^2g_{\mu\nu} + p_1^2p_{3\mu}p_{3\nu} + p_3^2p_{1\mu}p_{1\nu} - \left(p_1\cdot p_3\right)p_{1\mu}p_{3\nu} \\
& + p_1^2p_3^{\mu}p_2\nu - \left(p_1\cdot p_3\right)p_{1\mu}p_{2\nu} + p_3^2p_1^{\nu}p_2\mu - \left(p_1\cdot p_3\right)p_{2\mu}p_{3\nu}\Bigg] \\
=& \frac{1}{m_W^2}\int\frac{d^4k}{D_1D_2D_3}\Bigg[4\left(k_1\cdot k_2\right)k_{\mu}k_{\nu} + 2k^2k_{2\mu}k_{1\nu} \\
& - 2\left(k_{\mu}k_{1\nu}+k_{2\mu}k_{\nu}\right)\left(k\cdot\left(k_1+k_2\right)\right) + g_{\mu\nu}\left[-2k^2(k_1\cdot k_2)+(k\cdot(k_1+k_2))^2\right] \\
& + \left(k^2-\tfrac{1}{2}k_1\cdot k_2\right)\left[-g_{\mu\nu}\left(k\cdot\left(k_1-k_2\right)\right)+2\left(k_{\mu}k_{1\nu}-k_{2\mu}k_{\nu}\right)\right]\Bigg]
\end{aligned} \tag{3.21}$$

We can rewrite $k^2-\frac{1}{2}k_1\cdot k_2$ so that:

$$\begin{aligned}
k^2-\tfrac{1}{2}k_1\cdot k_2 &= k^2-\tfrac{1}{2}k_1\cdot k_2 - k\cdot(k_1-k_2) - m_W^2 + k\cdot(k_1-k_2) + m_W^2 \\
&= p_2^2 - m_W^2 + k\cdot(k_1-k_2) + m_W^2 = D_2 + k\cdot(k_1-k_2) + m_W^2
\end{aligned} \tag{3.22}$$

The (3.21) becomes:

$$\begin{aligned}
i\mathcal{M}_1^{(2)} =& \frac{1}{m_W^2}\int\frac{d^4k}{D_1D_2D_3}\Bigg[4\left(k_1\cdot k_2\right)k_{\mu}k_{\nu} + 2k^2k_{2\mu}k_{1\nu} - 4k_{\mu}k_{1\nu}\left(k\cdot k_2\right) \\
& - 4k_{2\mu}k_{\nu}\left(k\cdot k_1\right) + g_{\mu\nu}\left[-2k^2(k_1\cdot k_2)+4(k\cdot k_1)(k\cdot k_2)\right]\Bigg] \\
& + \frac{1}{m_W^2}\int\frac{d^4k}{D_1D_3}\left(-g_{\mu\nu}\left(k\cdot\left(k_1-k_2\right)\right)+2\left(k_{\mu}k_{1\nu}-k_{2\mu}k_{\nu}\right)\right) \\
& + \int\frac{d^4k}{D_1D_2D_3}\left(-g_{\mu\nu}\left(k\cdot\left(k_1-k_2\right)\right)+2\left(k_{\mu}k_{1\nu}-k_{2\mu}k_{\nu}\right)\right)
\end{aligned} \tag{3.23}$$

The third integral is $O\left(m_W^0\right)$, and we will analyze it later (we call it E). The second integral is odd in k ($D_1 \leftrightarrow D_3$ for $k \to -k$), so it vanishes. Thus we have shown that all the integrals with more-than-logarithmic divergences identically vanished, as we expected by gauge invariance. It is interesting now to evaluate the first integral (we call it F). It being only logarithmically divergent, we can shift the integration variable without adding any additional surface term[1]. We can combine the denominators with Feynman parametrization:

$$F = \frac{2}{m_W^2} \int d^4k \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \frac{\delta\left(1 - x_1 - x_2 - x_3\right)}{\left(D_1 x_1 + D_2 x_3 + D_3 x_2\right)^3} \left[\dots\right] \tag{3.24}$$

and, since $x_1 + x_2 + x_3 = 1$,

$$\begin{aligned} D_1 x_1 + D_2 x_3 + D_3 x_2 = \left(k - \tfrac{1}{2} k_1 \left(1 - 2x_1\right) + \tfrac{1}{2} k_2 \left(1 - 2x_2\right)\right)^2 \\ - m_W^2 + 2 x_1 x_2 \left(k_1 \cdot k_2\right) \end{aligned} \tag{3.25}$$

We can shift the integral to $l = k - \frac{1}{2} k_1 \left(1 - 2x_1\right) + \frac{1}{2} k_2 \left(1 - 2x_2\right)$. Integrating out x_3:

$$\begin{aligned} F = &\frac{2}{m_W^2} \int d^4 l \int_{\text{simplex}} dx_1 dx_2 \frac{1}{\left(l^2 - m_W^2 + 2x_1 x_2 \left(k_1 \cdot k_2\right)\right)^3} \\ &\times \Big[4 l^\alpha l^\beta \left(g_{\mu\alpha} g_{\nu\beta} \left(k_1 \cdot k_2\right) - g_{\mu\alpha} k_{1\nu} k_{2\beta} - g_{\nu\beta} k_{1\alpha} k_{2\mu} + g_{\mu\nu} k_{1\alpha} k_{2\beta}\right) \\ &+ 2 l^2 \left(k_{1\nu} k_{2\mu} - g_{\mu\nu} \left(k_1 \cdot k_2\right)\right)\Big] \end{aligned} \tag{3.26}$$

where $\int_{\text{simplex}} dx_1 dx_2$ is a shortcut for $\int_0^1 dx_1 \int_0^{1-x_1} dx_2$.

The crucial point now is the choice to integrate symmetrically. Since we stick to four-dimensional calculations, we have $l_\alpha l_\beta \to \frac{1}{4} l^2 g_{\alpha\beta}$ [2]. If we are able to, the

[1] Actually, if we consider the translation of an integral $\Delta(a) = \int^\Lambda d^n x \left[f(x+a) - f(x)\right]$, and expand in powers of a, we get $\Delta(a) = \int^\Lambda d^n x \left[a_\mu \partial^\mu f + \frac{1}{2} a_\mu a_\nu \partial^\mu \partial^\nu f + \dots\right] = a_\mu \oint_{S_\Lambda^n} d\sigma^\mu \left[f + \frac{1}{2} a_\nu \partial^\nu f + \dots\right]$. If the integral is logarithmically divergent, we have $f(\Lambda) \approx O\left(1/\Lambda^n\right)$, and $\partial^i f(\Lambda) \approx O\left(1/\Lambda^{n+i}\right)$. Since the superficial integral provides a factor of Λ^{n-1}, the translation is $O(1/\Lambda) \to 0$. On the other hand, a linear divergent integral has a finite surface term. This is, for example, a way to show the arising of axial anomaly [27]

[2] If we take indeed all the external momenta out of the integral, we have a tensor expression $I_{\alpha\beta}$ which can depend only on constant tensors. Since the only two-index constant tensors is $g_{\alpha\beta}$, it must be $I_{\alpha\beta} = I g_{\alpha\beta}$. By saturating both sides with $g^{\alpha\beta}$, we have $I_{\alpha\beta} g^{\alpha\beta} \to nI$ (where n is the dimension of the space), and inside the integral $l^2 g_{\alpha\beta} \to n l^2$ and $l_\alpha l_\beta \to l^2$, then we can solve the equality with respect to I. We have the same result if we substitute $l_\alpha l_\beta \to l^2 g_{\alpha\beta}/n$. In our actual calculations, we stick to a four-dimensional space, and so $l_\alpha l_\beta \to l^2 g_{\alpha\beta}/4$.

integral vanishes. The same happens to the correspondent integral of $\mathcal{M}_2$.

We have the terms m_W^0 left. We must choose the non-longitudinal part of the propagators in $\mathcal{M}_1$,$\mathcal{M}_2$,$\mathcal{M}_3$; and add the C and E integrals resp. from (3.15) (3.23). We have only

$$\begin{aligned} i\mathcal{M}^{(0)} &= \left(i\mathcal{M}_1^{(0)} + C_1^3 + C_1^1 + E_1\right) + \left(i\mathcal{M}_2^{(0)} + C_2^3 + C_2^1 + E_2\right) + i\mathcal{M}_3^{(0)} \\ &= \int d^4k \frac{1}{D_1 D_2 D_3} \Big[g_{\mu\nu} \left(-6k^2 + 6(k\cdot k_1) - 6(k\cdot k_2) - 9(k_1\cdot k_2) + 6m_W^2\right) \\ &\quad + 24k_\mu k_\nu + 6k_{2\mu}k_{1\nu} - 12k_\mu k_{1\nu} + 12k_{2\mu}k_\nu \Big] \end{aligned} \tag{3.27}$$

The integral is still logarithmically divergent. We combine the denominators and shift the integral as in (3.24); since the denominator is now even in l, we can drop the terms odd in l in the numerator, and get:

$$\begin{aligned} i\mathcal{M}^{(0)} =& 2\int d^4l \int_{\text{simplex}} dx_1 dx_2 \frac{1}{\left(l^2 - m_W^2 + 2x_1x_2\,(k_1\cdot k_2)\right)^3} \\ &\times \Big[24 l_\mu l_\nu + 12\,(1-2x_1x_2)\,k_{1\nu}k_{2\mu} \\ &+ 6m_W^2 g_{\mu\nu} - 12(k_1\cdot k_2)\,(1-x_1x_2)\,g_{\mu\nu} - 6l^2 g_{\mu\nu}\Big] \end{aligned} \tag{3.28}$$

Again, we integrate symmetrically: $l_\mu l_\nu \to \frac{1}{4} l^2 g_{\mu\nu}$, and get

$$\begin{aligned} i\mathcal{M}^{(0)} =& 2\int d^4l \int_{\text{simplex}} dx_1 dx_2 \frac{1}{\left(l^2 - m_W^2 + 2x_1x_2\,(k_1\cdot k_2)\right)^3} \\ &\times 6\Big[2\,(1-2x_1x_2)\,k_{1\nu}k_{2\mu} \\ &+ m_W^2 g_{\mu\nu} - 2(k_1\cdot k_2)\,(1-x_1x_2)\,g_{\mu\nu}\Big] \end{aligned} \tag{3.29}$$

The divergent terms disappear, so we can integrate without problems. We restore the constant $e^2 g\, m_W/(2\pi)^4$:

$$\begin{aligned} \mathcal{M} =& \frac{e^2 g\, m_W}{(4\pi)^2} \int_{\text{simplex}} dx_1 dx_2 \frac{1}{m_W^2 - 2x_1x_2\,(k_1\cdot k_2)} \\ &\times 6\Big[-2\,(1-2x_1x_2)\,k_{1\nu}k_{2\mu} \\ &- m_W^2 g_{\mu\nu} + 2(k_1\cdot k_2)\,(1-x_1x_2)\,g_{\mu\nu}\Big] \end{aligned} \tag{3.30}$$

We have obtained a non-gauge invariant result (it does not respect the Ward identity, since $\mathcal{M}^{\mu\nu}k_{1\mu} \neq 0$). We solve this problem with Dyson subtraction [28, 29]:

$$\begin{aligned}\mathcal{M} \to \mathcal{M} - \mathcal{M}\Big|_{k_1=k_2=0} =& \frac{e^2 g\, m_W}{(4\pi)^2} \left[(k_1 \cdot k_2) g_{\mu\nu} - k_{1\nu}k_{2\mu}\right] \\ & \times \int_{\text{simplex}} dx_1 dx_2 \frac{12\,(1-2x_1x_2)}{m_W^2 - 2x_1x_2\,(k_1\cdot k_2)}\end{aligned} \tag{3.31}$$

The integral on Feynman parameters can be evaluated (see Appendix A):

$$\mathcal{M} = \frac{3e^2 g}{(4\pi)^2\, m_W} \left[(k_1 \cdot k_2) g_{\mu\nu} - k_{1\nu}k_{2\mu}\right] \left(\tau^{-1} + \left(2\tau^{-1} - \tau^{-2}\right)\, f(\tau)\right) \tag{3.32}$$

where $\tau = \frac{m_H^2}{4m_W^2}$ and

$$f(\tau) = \begin{cases} \arcsin^2(\sqrt{\tau}) & \text{for} \quad \tau \leq 1 \\ -\dfrac{1}{4}\left[\ln \dfrac{1+\sqrt{1-\tau^{-1}}}{1-\sqrt{1-\tau^{-1}}} - i\pi\right]^2 & \text{for} \quad \tau > 1 \end{cases} \tag{3.33}$$

3.2 Properties of the amplitude

We quote the old result of Ref. [4]:

$$\mathcal{M} = \frac{e^2 g}{(4\pi)^2\, m_W} \left[(k_1 \cdot k_2) g_{\mu\nu} - k_{1\nu}k_{2\mu}\right] \left(2 + 3\tau^{-1} + 3\left(2\tau^{-1} - \tau^{-2}\right)\, f(\tau)\right) \tag{3.34}$$

The two solutions have a qualitative different behaviour in the limit $\tau \to \infty$ (i.e., a heavy Higgs boson or a light W): while Gastmans *et al.*'s amplitude (3.32) vanishes, the old result above (3.34) does not because of an additional term which does not depend on τ.

The latter amplitude has to be compared with the $H \to \gamma\gamma$ decay via fermion loop (Figure 3.2):

$$\mathcal{M} = -\frac{e^2 g}{(4\pi)^2\, m_W} \left[(k_1 \cdot k_2) g_{\mu\nu} - k_{1\nu}k_{2\mu}\right] \sum_f 2N_c e_f\, \tau_f^{-1} \left(1 + \left(1 - \tau_f^{-1}\right)\, f(\tau_f)\right) \tag{3.35}$$

with $\tau_f = \frac{m_H^2}{4m_f^2}$, e_f is the electric charge, and N_c is the number of colors (1 for leptons and 3 for quarks). It is easy to see that the fermion loop amplitude vanishes in the limit $\tau_f \to \infty$, and this explains why light fermions contribute negligibly to the amplitude. Moreover, in this amplitude we have neither a gauge to choose, nor unphysical particles; the highest divergent integral vanishes because it containt a trace of an odd number of gamma matrices, and the remaining logarithmically divergent integrals can be evaluated either with dimensional regularization, or with the methods above, getting to the same result (3.35), which vanishes for $\tau_f \to \infty$. This result is therefore "stable", meaning that there is no mathematical or physical quibble which could challenge the calculation.

Gastmans *et al.* argue that the W loop amplitude should have the same "decoupling" behaviour of the fermion loop in the limit $\tau \to \infty$. Furthermore, they invoke the decoupling theorem [5] to say that the Higgs boson must cease interaction with other particles when its mass grows arbitrarily large; since the old amplitude (3.34) does not decouple, it must be wrong.

The origin of the mistake is traced back to dimensional regularization, and in particular to integrals as in Eqs. (3.26) and (3.29):

$$I_{\mu\nu} = \int d^4l \, \frac{g_{\mu\nu} l^2 - 4 l_\mu l_\nu}{\left(l^2 - M^2 + i\epsilon\right)^3} \tag{3.36}$$

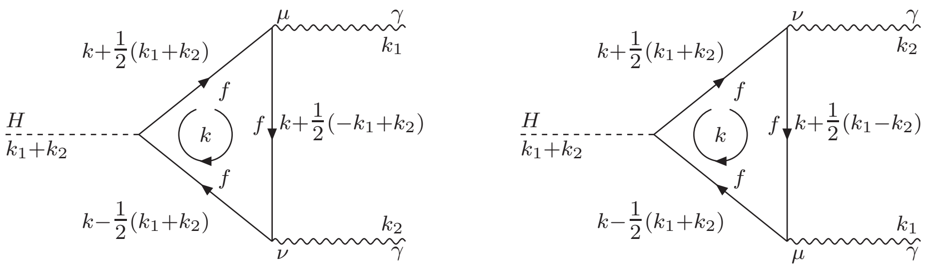


FIGURE 3.2: The one-loop diagrams with virtual fermions that contribute to the amplitude for $H \to \gamma\gamma$.

If we have recourse to dimensional regularization, we can substitute $l_\mu l_\nu \to l^2 g_{\mu\nu}/n$, and so

$$
\begin{aligned}
I^{\mathrm{DR}}_{\mu\nu} &= \int d^n l \, \frac{g_{\mu\nu} l^2 - \frac{4}{n} g_{\mu\nu} l^2}{\left(l^2 - M^2 + i\epsilon\right)^3} = i M^{n-4} \, g_{\mu\nu} \frac{n-4}{n} \frac{\pi^{n/2}}{2\Gamma(n/2)} \Gamma\left(\frac{n}{2}+1\right) \Gamma\left(2-\frac{n}{2}\right) \\
&\xrightarrow{n \to 4} -i g_{\mu\nu} \frac{\pi^2}{2}
\end{aligned}
\tag{3.37}
$$

If we stick instead to four-dimensional calculations, we get $I_{\mu\nu} = 0$ by symmetric integration; moreover, with dimensional regularization, the $I_{\mu\nu}$ tensor looses the property to be traceless. Hence, the tensor $I^{\mathrm{DR}}_{\mu\nu}(n)$ in the limit $n \to 4$ is different from the four dimensional $I_{\mu\nu}$; in other words, $I^{\mathrm{DR}}_{\mu\nu}(n)$ has a discountinuity in $n = 4$, and so the main hypothesis behind dimensional regularization (the analicity in n) is spoiled.

Chapter 4

Why decoupling?

4.1 The Equivalence theorem

Just two weeks after the publication of Gastmans *et al.* results, an instant-paper [6] by the authors of the old calculations [4] appeared, rejecting the new result. Their argument is based on the Goldstone bosons Equivalence theorem [30–32], and has been drawn on by some other papers [7, 10].

In general, the theorem describes the behaviour of gauge bosons in a spontaneously broken theory in the limit of vanishing mass of gauge bosons. In a popular picture, gauge bosons "swallow" the Goldstone bosons to acquire a mass and, with it, a longitudinal polarization. However, since the mass of a gauge boson is proportional to gauge coupling, if we take the limit of vanishing mass we must recover the original scalar theory without gauge fields; in other words, the gauge bosons "spit out" the Goldstone bosons and *then* decouple, leaving a non-vanishing interaction with the Goldstone bosons.

We want to show this mechanism at the tree level. Technically, the interaction term HWW in unitary gauge is:

$$\mathcal{L}_{HWW} = 2\frac{m_W^2}{v} H\, W_\mu^\dagger W^\mu \tag{4.1}$$

It would seem that the vertex vanishes for $m_W \to 0$. If we explicit instead the polarization vectors:

$$\begin{aligned} \epsilon^\mu_\pm &= \frac{1}{\sqrt{2}}(0, 1, \pm i, 0) \\ \epsilon^\mu_L &= \frac{1}{m_W}(k, 0, 0, E) = \frac{k^\mu}{m_W} + O\left(\frac{m_W}{E}\right) \end{aligned} \tag{4.2}$$

The "miracle" of longitudinal polarization is that it tends to be parallel to k^μ in the limit $m_W \to 0$. If we separate the longitudinal term and express the vertex in terms of a scalar field ϕ, we have:

$$\mathcal{L}_{HWW} = 2\frac{m_W^2}{v} H \left(W^T\right)^\dagger_\mu \left(W^T\right)^\mu + 2\frac{H}{v}\partial_\mu\phi^\dagger\partial^\mu\phi \tag{4.3}$$

The transverse polarizations actually decouple, whereas the longitudinal polarization does not! At tree level, we can use the equations of motion to obtain the vertex of the unbroken scalar theory:

$$\begin{aligned} \mathcal{L}^{m_W\to 0}_{HWW} &= 2\frac{H}{v}\partial_\mu\phi^\dagger\partial^\mu\phi = 2\frac{H}{v}\left[\partial^2\left(\phi^\dagger\phi\right) - \left(\partial^2\phi^\dagger\right)\phi\right] \\ &= 2\frac{H}{v}\partial^2\left(\phi^\dagger\phi\right) = 2\frac{\partial^2 H}{v}\phi^\dagger\phi = -2\frac{m_H^2}{v} H\phi^\dagger\phi = -\lambda v\, H\phi^\dagger\phi \end{aligned} \tag{4.4}$$

We see indeed that the coupling λv does not depend on g, and so does not vanish when $m_W \to 0$. Following Marciano *et al.* [7], we perform the explicit calculations in scalar theory. The three diagrams are drawn in Figure 4.1. With the conventions used in the previous chapter, we have

$$i\mathcal{M}_1^{\mathrm{GB}} = \frac{e^2\lambda v}{(2\pi)^4}\int d^4k \frac{(p_1+p_2)_\mu (p_2+p_3)_\nu}{D_1 D_2 D_3} \tag{4.5}$$

$$i\mathcal{M}_2^{\mathrm{GB}} = i\mathcal{M}_1^{\mathrm{GB}}\left(\mu \leftrightarrow \nu,\, k_1 \leftrightarrow k_2\right) \tag{4.6}$$

$$i\mathcal{M}_3^{\mathrm{GB}} = \frac{e^2\lambda v}{(2\pi)^4}\int d^4k \frac{2g_{\mu\nu}}{D_1 D_3} \tag{4.7}$$

where $D_i = p_i^2$ because the Goldstone bosons are massless. The integrals are logarithmically divergent at worst, so we can combine the denominators with Feynman parametrization and shift the integrals:

$$
\begin{aligned}
i\mathcal{M}^{\mathrm{GB}} =& \frac{e^2\lambda v}{(2\pi)^4}\, 2\int_{\text{simplex}} dx_1 dx_2 \int d^4 l \times \left[\frac{(4l_\mu l_\nu - l^2 g_{\mu\nu})}{\left(l^2 + 2x_1x_2\,(k_1\cdot k_2)\right)^3} \right. \\
&\left. + \frac{4x_1x_2\left(\frac{1}{2}\,(k_1\cdot k_2)\,g_{\mu\nu} - k_{1\nu}k_{2\mu}\right)}{\left(l^2 + 2x_1x_2\,(k_1\cdot k_2)\right)^3} \right]
\end{aligned} \tag{4.8}
$$

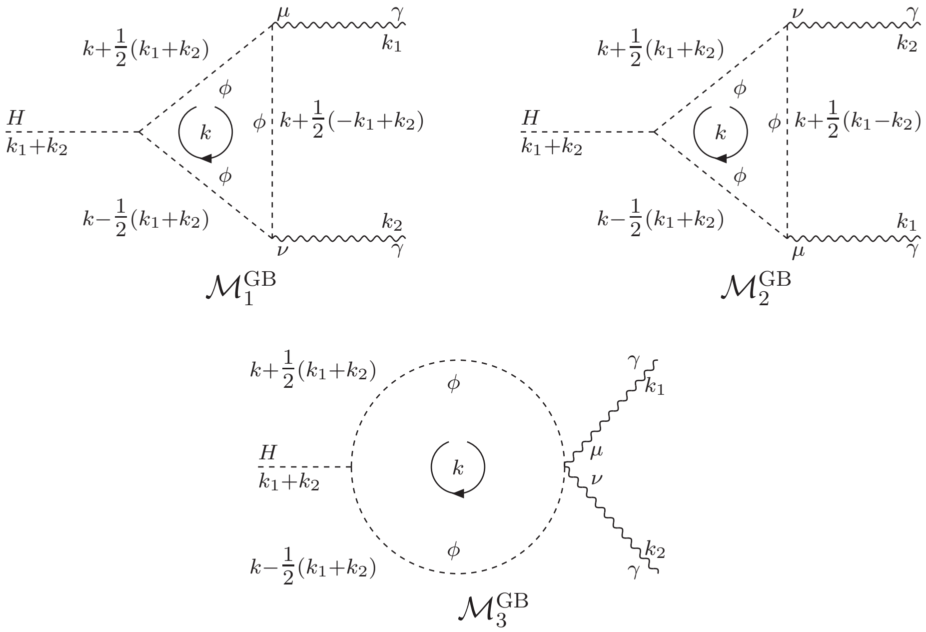


FIGURE 4.1: The one-loop diagrams with only Goldstone bosons, which give the leading contribution in m_W/m_H to the amplitude for $H \to \gamma\gamma$.

The integral in the first line is still divergent. If we stick to Gastmans *et al.*'s procedure, the divergent part of the integral vanishes by a four-dimensional symmetric integration. We can therefore integrate on l:

$$
\begin{aligned}
\mathcal{M}^{\mathrm{GB}} &= \frac{e^2\lambda v}{(4\pi)^2}\, 2\int_{\text{simplex}} dx_1 dx_2 \frac{2x_1x_2\left(\frac{1}{2}\,(k_1\cdot k_2)\,g_{\mu\nu} - k_{1\nu}k_{2\mu}\right)}{2x_1x_2\,(k_1\cdot k_2)} \\
&= \frac{e^2\lambda v}{(4\pi)^2} \frac{\frac{1}{2}\,(k_1\cdot k_2)\,g_{\mu\nu} - k_{1\nu}k_{2\mu}}{k_1\cdot k_2}
\end{aligned} \tag{4.9}
$$

Gauge invariance must be restored again with Dyson subtraction. Unfortunately, we cannot simply impose $k_1 = k_2 = 0$ because of the denominator. The coefficient of $g_{\mu\nu}$ simplifies the denominator, while the $k_{1\nu}k_{2\mu}$ term has no clear limit. If we accept that $k_{1\nu}k_{2\mu}/\left(k_1 \cdot k_2\right) \to g_{\mu\nu}$, we can properly perform the subtraction[1]:

$$\begin{aligned}\mathcal{M}^{\mathrm{GB}} \to \mathcal{M}^{\mathrm{GB}} - \mathcal{M}^{\mathrm{GB}}\Big|_{k_1=k_2=0} &= \frac{e^2\lambda v}{\left(4\pi\right)^2}\left(\frac{\frac{1}{2}\left(k_1 \cdot k_2\right) g_{\mu\nu} - k_{1\nu}k_{2\mu}}{k_1 \cdot k_2} + \frac{1}{2}g_{\mu\nu}\right) \\ &= \frac{e^2\lambda v}{\left(4\pi\right)^2\left(k_1 \cdot k_2\right)}\left[\left(k_1 \cdot k_2\right) g_{\mu\nu} - k_{1\nu}k_{2\mu}\right]\end{aligned} \tag{4.10}$$

We can express the constants in terms of m_H and m_W:

$$\mathcal{M}^{\mathrm{GB}} = \frac{e^2 g}{\left(4\pi\right)^2 m_W}\left[\left(k_1 \cdot k_2\right) g_{\mu\nu} - k_{1\nu}k_{2\mu}\right] \times 2 \tag{4.11}$$

We remark that, in a gauge invariant regularization, the divergent integral in (4.8) would provide just the same additional term as Dyson subtraction, so that the result would be automatically transverse. The factor 2 is exactly the non-decoupling term of Eqn. (3.34).

To resume, we calculated the $H \to \gamma\gamma$ amplitude via the Equivalence theorem; we followed Gastmans *et al.* prescriptions to deal with divergent integrals: we stuck to four-dimensional integration, and used Dyson subtraction to recover the invariance on the final amplitude. We found a non-zero result which agrees with the old result (3.34) in the limit $m_W \to 0$, and is at odds with the new one by Gastmans *et al.*. The idea that a different result is the sign of a failure of dimensional regularization is confuted.

4.2 The decoupling theorem

Hence, the Equivalence theorem suggests that a heavy Higgs boson does not decouple from light W bosons in the limit $m_W/m_H \to 0$. However, Gastmans *et al.* used the decoupling theorem to justify that the amplitude must vanish in this limit. Is there a failure of the decoupling theorem?

In their original paper, Appelquist and Carazzone [5] formulated the decoupling theorem in a gauge theory with heavy fermions: in 1-PI diagrams with gauge bosons on the external legs, the interactions with heavy fermions can be reabsorbed

[1] Otherwise, we could simply remark that the $k_{1\nu}k_{2\mu}$ term is finite in the intermediate calculations; thus the action of any regulator or subtraction can only tune the $g_{\mu\nu}$ coefficient to match the $k_{1\nu}k_{2\mu}$ one.

into an effective gauge coupling, provided that the renormalization scale is much smaller than the fermion masses. In other words, all the low-energy amplitudes computed in the theory without the heavy degrees of freedom coincide (up to a redefinition of the renormalization parameters) with those in the full theory in the limit where the masses of the heavy part of the spectrum are pushed to infinity. The original theorem has been generalized and applied to many theories with different scales of mass, provided the low-energy theory remains renormalizable after integrating out the heavy degrees of freedom. Indeed, we have some examples in the Standard Model of non-decoupling contributions by the top quark, because in the limit $m_t \to \infty$ the isospin doublet with the bottom quark is broken, and the resulting theory is not renormalizable.

That said, it is clear why the theorem cannot be applied in the limit $m_H \to \infty$ as Gastmans *et al.* said: the heavy degree of freedom is on an external leg, and therefore it cannot be absorbed into an effective renormalization coupling as the theorem requires.

4.3 The limit $m_H \to 0$

Nevertheless, it is interesting to explore the opposite limit $m_H/m_W \to 0$. One of the initial critics by Gastmans *et al.* was indeed that the old calculations of Refs. [3, 4] were performed in the limit of light Higgs boson, and lose meaning in the opposite limit where the decoupling shold have occurred. However, the new result is quite different also in the limit of light Higgs boson:

$$\begin{aligned}
\mathcal{M}_{\tau\to 0} &= \frac{e^2 g}{(4\pi)^2 m_W} \left[(k_1 \cdot k_2)\, g_{\mu\nu} - k_{1\nu} k_{2\mu}\right] \epsilon_1^\mu \epsilon_2^\nu \times 7 \quad \text{for the old result (3.34)} \\
\mathcal{M}_{\tau\to 0} &= \frac{e^2 g}{(4\pi)^2 m_W} \left[(k_1 \cdot k_2)\, g_{\mu\nu} - k_{1\nu} k_{2\mu}\right] \epsilon_1^\mu \epsilon_2^\nu \times 5 \quad \text{for the new result (3.32)}
\end{aligned} \tag{4.12}$$

We briefly summarize the original argument by Shifman *et al.* [4]: it is based on renormalization group considerations, and so its conclusions should be quite general.

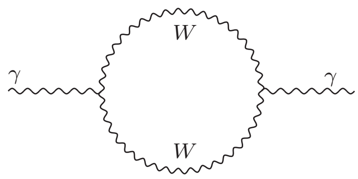


FIGURE 4.2: One-loop vacuum polarization via W's. If we add a classical H field, the only effect is the shifting of the mass of W. We can therefore extract an effective $H\gamma\gamma$ vertex from this diagram.

Let us start from the vacuum polarization in the Weinberg-Salam theory. We can write an effective lagrangian for the loop in Figure 4.2:

$$\mathcal{L}^{\text{eff}}_{\gamma\gamma} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sum_i \frac{-b_i\, e^2}{(4\pi)^2} \ln \frac{\Lambda}{M_i} \tag{4.13}$$

where the sum runs over all charged fields with mass M_i, the b_i are the one-loop coefficients of the β function, and Λ is an ultraviolet cutoff. In a pure Yang-Mills $SU(N)$ theory, we have $b = 11\,N/3$. For a massive W, we must consider the contribution of the longitudinal polarization, i.e. an additional scalar particle which adds a $-1/3$ value.

$$b_W = \frac{11}{3} N - \frac{1}{3} = \frac{22}{3} - \frac{1}{3} = 7 \tag{4.14}$$

What happens to vacuum polarization if we add a light Higgs field? The interaction lagrangian is:

$$\mathcal{L}^{\text{int}}_{HWW} = g\, m_W\, W^\dagger_\mu W^\mu\, H \tag{4.15}$$

If H is a light field, we can approximate it with a constant-valued classical field. In this way, it only contributes with a shift of m_W:

$$m_W^2 \rightarrow m_W^2 + g\, m_W\, H \tag{4.16}$$

If we insert it into (4.13) and take only the first-order term in H, we have:

$$\mathcal{L}^{\text{eff}}_{H\gamma\gamma} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \frac{-7\, e^2}{(4\pi)^2} \left(-\frac{g\, H}{m_W} \right) = \frac{e^2 g}{(4\pi)^2\, m_W} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \times 7 \tag{4.17}$$

This corresponds to a tree-level $H \to \gamma\gamma$ effective amplitude:

$$\mathcal{M}^{\text{eff}} = \frac{e^2 g}{(4\pi)^2 m_W} \left[(k_1 \cdot k_2)\, g_{\mu\nu} - k_{1\nu} k_{2\mu} \right] \epsilon_1^\mu \epsilon_2^\nu \times 7 \tag{4.18}$$

This actually agrees with the limit $\tau \to 0$ of the old result (3.34), whereas Gastmans *et al.* have a factor 5 in the place of the factor 7. Again, it seems that the result by Gastmans *et al.* is at odds with renormalization group equations.

Just a few words about the decoupling theorem in the limit $m_W \to \infty$. In a naive way, we could simply accept that $\mathcal{M} \to 0$ in the limit $m_W \to \infty$ because of the m_W in the denominator. However, in the Weinberg-Salam theory, $m_W = \frac{1}{2} g\, v$ and $m_H = \lambda v$. Pushing the VEV to infinity would scale all masses to infinity without modifying their ratios. On the other hand, if we want to push $m_W \to \infty$, we cannot push $g \to \infty$ because the perturbative framework would be spoiled (moreover, $\mathcal{M}$ does not depend on g if we explicit m_W). To explore the limit $m_H/m_W \to 0$, we can though take $\lambda \to 0$. If we take indeed the squared matrix, and explicit the dependence on λ, g and v, we have:

$$|\mathcal{M}|^2 = \frac{v^2\, e^4}{(4\pi)^4} \left(49 + O(\lambda) \right) \frac{\lambda^4}{2} \xrightarrow{\lambda \to 0} 0 \tag{4.19}$$

The application of the decoupling theorem turns out to be more cumbersome than Gastmans *et al.* thought.

Chapter 5

R_ξ gauges and cutoffs

In the previous chapter we showed that the expression of the amplitude $H \to \gamma\gamma$ by Gastmans *et al.* does not fulfill the Equivalence theorem in the limit $m_W << m_H$, and is not in agreement with the analysis of Renormalization group equations in the limit $m_H << m_W$. However, we have not yet shown where the weak point in Gastmans *et al.*'s argument is. We now want to repeat the calculation in a renormalizable gauge, to see whether the problem resides in the highly divergent integrals which appear in unitary gauge. In particular, we choose 't Hooft-Feynman gauge ($\xi = 1$), because in this gauge the gauge bosons, the Goldstone bosons and the Faddeev-Popov ghosts have the same mass, and this leads to a simplification in Feynman parameters.

5.1 The evaluation in 't Hooft-Feynman gauge

In renormalizable gauges, we have a lot of different diagrams because we have to deal also with Goldstone bosons and ghosts. We show them in Figure 5.1. To simplify the expression of matrix elements, we will omit the constant factor $e^2 g m_W / (2\pi)^4$ and the polarization vectors. About the ghosts, there is an additional minus for the fermionic trace. We have 14 diagrams and their crossed counterparts:

$$\begin{aligned}
i\mathcal{M}_{WWW} =& \int d^4k \frac{g^{\alpha\beta}\, g^{\rho\sigma}\, g^{\gamma}_{\alpha}}{D_1 D_2 D_3} \\
&\times \left[-\left(k + \tfrac{3}{2}k_1 + \tfrac{1}{2}k_2\right)_\rho g_{\beta\mu} - \left(k - \tfrac{3}{2}k_1 + \tfrac{1}{2}k_2\right)_\beta g_{\mu\rho} + (2k + k_2)_\mu\, g_{\rho\beta} \right] \\
&\times \left[-\left(k - \tfrac{1}{2}k_1 - \tfrac{3}{2}k_2\right)_\sigma g_{\gamma\nu} - \left(k - \tfrac{1}{2}k_1 + \tfrac{3}{2}k_2\right)_\gamma g_{\nu\sigma} + (2k - k_1)_\nu\, g_{\sigma\gamma} \right]
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
i\mathcal{M}_{\phi WW} =& \frac{1}{2} \int d^4k \frac{g^{\rho\sigma}\, g^{\gamma\alpha}\, g_{\rho\mu}}{D_1 D_2 D_3} \left(k + \tfrac{3}{2}k_1 + \tfrac{3}{2}g_2\right)_\alpha \\
&\times \left[-\left(k - \tfrac{1}{2}k_1 - \tfrac{3}{2}k_2\right)_\sigma g_{\gamma\nu} - \left(k - \tfrac{1}{2}k_1 + \tfrac{3}{2}k_2\right)_\gamma g_{\nu\sigma} + (2k - k_1)_\nu\, g_{\sigma\gamma} \right]
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
i\mathcal{M}_{WW\phi} =& \frac{1}{2} \int d^4k \frac{g_{\sigma\nu}\, g^{\beta\alpha}\, g^{\rho\sigma}}{D_1 D_2 D_3} \left(k - \tfrac{3}{2}k_1 - \tfrac{3}{2}g_2\right)_\alpha \\
&\times \left[-\left(k + \tfrac{3}{2}k_1 + \tfrac{1}{2}k_2\right)_\rho g_{\beta\mu} - \left(k - \tfrac{3}{2}k_1 + \tfrac{1}{2}k_2\right)_\beta g_{\mu\rho} + (2k + k_2)_\mu\, g_{\rho\beta} \right]
\end{aligned} \tag{5.3}$$

$$i\mathcal{M}_{W\phi W} = - m_W^2 \int d^4k \frac{g_{\beta\mu}\, g_{\gamma\nu}\, g^{\alpha\beta}\, g^{\gamma}_{\alpha}}{D_1 D_2 D_3} \tag{5.4}$$

$$i\mathcal{M}_{W\phi\phi} = \frac{1}{2} \int d^4k \frac{g_{\beta\mu}\, g^{\beta\alpha}}{D_1 D_2 D_3} \left(k - \tfrac{3}{2}k_1 - \tfrac{3}{2}g_2\right)_\alpha (2k - k_1)_\nu \tag{5.5}$$

$$i\mathcal{M}_{\phi\phi W} = \frac{1}{2} \int d^4k \frac{g_{\gamma\nu}\, g^{\alpha\gamma}}{D_1 D_2 D_3} \left(k + \tfrac{3}{2}k_1 + \tfrac{3}{2}g_2\right)_\alpha (2k + k_2)_\mu \tag{5.6}$$

$$i\mathcal{M}_{\phi W\phi} = - \frac{m_H^2}{2} \int d^4k \frac{g_{\nu\sigma}\, g_{\mu\rho}\, g^{\rho\sigma}}{D_1 D_2 D_3} \tag{5.7}$$

$$i\mathcal{M}_{\phi\phi\phi} = \frac{m_H^2}{2m_W^2} \int d^4k \frac{1}{D_1 D_2 D_3} (2k + k_2)_\mu (2k - k_1)_\nu \tag{5.8}$$

$$i\mathcal{M}_{WW} = - \int d^4k \frac{g^{\alpha\beta}\, g^{\gamma}_{\alpha}}{D_1 D_3} \left(2g_{\mu\nu}g_{\beta\gamma} - g_{\mu\beta}g_{\nu\gamma} - g_{\mu\gamma}g_{\nu\beta}\right) \tag{5.9}$$

$$i\mathcal{M}_{\phi\phi} = - \frac{m_H^2}{2m_W^2} \int d^4k \frac{2g_{\mu\nu}}{D_1 D_3} \tag{5.10}$$

$$i\mathcal{M}_{W\phi} = i\mathcal{M}_{\phi W} = -\frac{1}{2} \int d^4k \frac{g_{\mu\beta}\, g_{\nu\gamma}\, g^{\beta\gamma}}{D_2 D_3} \tag{5.11}$$

$$i\mathcal{M}_{\eta^+} = i\mathcal{M}_{\eta^-} = -\frac{1}{2} \int d^4k \frac{1}{D_1 D_2 D_3} \left(k - \tfrac{1}{2}k_1 + \tfrac{1}{2}k_2\right)_\mu \left(k - \tfrac{1}{2}k_1 - \tfrac{1}{2}k_2\right)_\nu \tag{5.12}$$

Since we have only logarithmic divergences, we can proceed as in Sec. 3.1: we take all fractions to the least common denominator, combine the denominators with

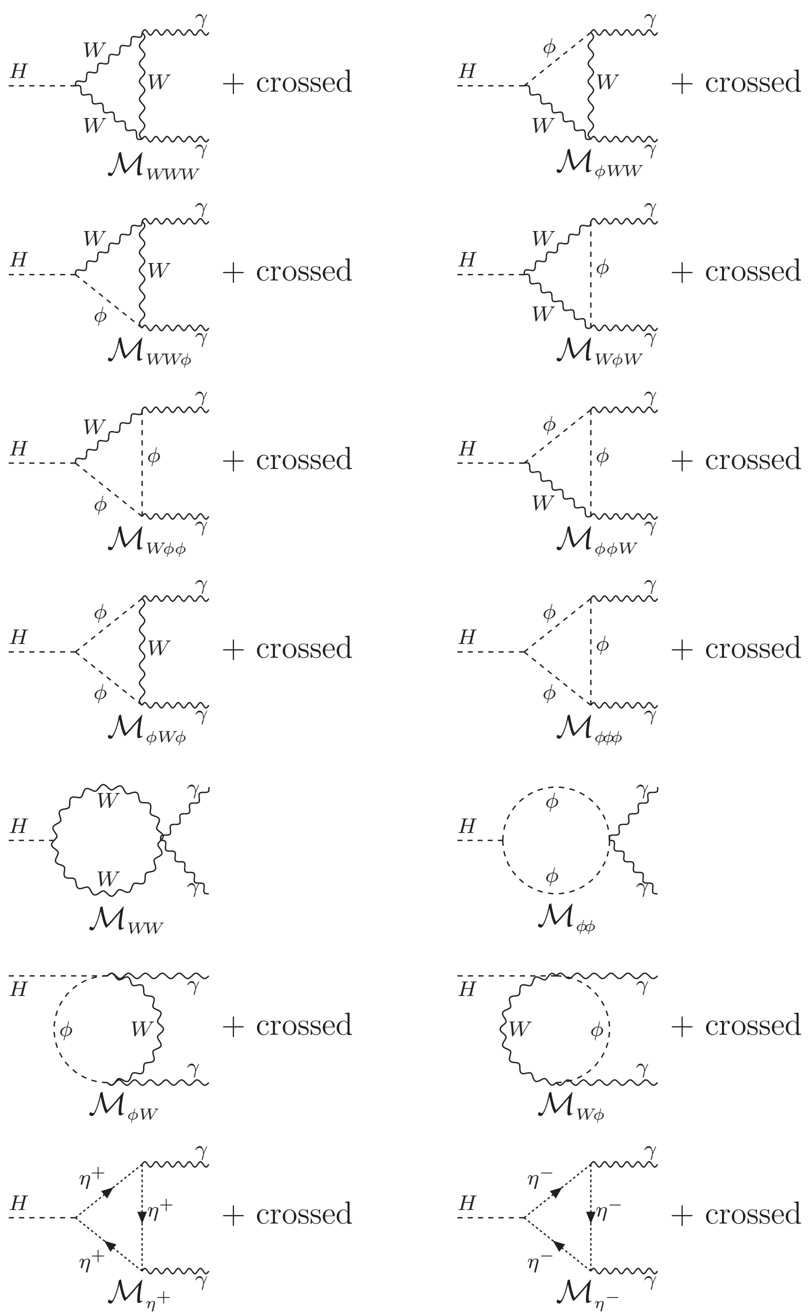


FIGURE 5.1: The one-loop diagrams in R_ξ gauge that contribute to the amplitude for $H \to \gamma\gamma$. The ϕ are the would-be Goldstone bosons, and the η the Faddeev-Popov ghosts.

Feynman parameters, and shift the integrals to

$$l = k - \tfrac{1}{2}k_1\,(1 - 2x_1) + \tfrac{1}{2}k_2\,(1 - 2x_2) \tag{5.13}$$

Moreover, we set $k_1^2 = k_2^2 = 0$ and $k_{1\mu} = k_{2\nu} = 0$, since we left out the contraction with the polarization vectors $\epsilon_1^\mu\,\epsilon_2^\nu$. Finally, we drop all odd terms in l.

$$\left(\text{omitted } \frac{e^2 g m_W}{(2\pi)^4}\epsilon_1^\mu\epsilon_2^\nu \; 2\int_{\text{simplex}} dx_1 dx_2 \int d^4k \frac{1}{\left(l^2 - m_W^2 + 2x_1x_2\left(k_1\cdot k_2\right)\right)^3}\right)$$

$$\begin{aligned}
i\mathcal{M}_{WWW} =& k_{1\nu}k_{2\mu}\left(4 + x_2 + x_1 - 10x_1x_2\right) \\
&+ k_1\cdot k_2\, g_{\mu\nu}\left(-5 + x_2 + x_1 - 4x_1x_2\right) + 10 l_\mu l_\nu + 2l^2\, g_{\mu\nu} && (5.14)\\
i\mathcal{M}_{\phi WW} =& k_{1\nu}k_{2\mu}\left(2 - x_2 - 2x_1 - \frac{1}{2}x_1x_2\right) \\
&+ k_1\cdot k_2\, g_{\mu\nu}\left(-1 - x_2 + x_1 + x_1x_2\right) + \frac{1}{2}l_\mu l_\nu - \frac{1}{2}l^2\, g_{\mu\nu} && (5.15)\\
i\mathcal{M}_{WW\phi} =& k_{1\nu}k_{2\mu}\left(2 - 2x_2 - x_1 - \frac{1}{2}x_1x_2\right) \\
&+ k_1\cdot k_2\, g_{\mu\nu}\left(-1 + x_2 - x_1 + x_1x_2\right) + \frac{1}{2}l_\mu l_\nu - \frac{1}{2}l^2\, g_{\mu\nu} && (5.16)\\
i\mathcal{M}_{W\phi W} =& - m_W^2 g_{\mu\nu} && (5.17)\\
i\mathcal{M}_{W\phi\phi} =& k_{1\nu}k_{2\mu}\left(2x_1 - x_1x_2\right) + l_\mu l_\nu && (5.18)\\
i\mathcal{M}_{\phi\phi W} =& k_{1\nu}k_{2\mu}\left(2x_2 - x_1x_2\right) + l_\mu l_\nu && (5.19)\\
i\mathcal{M}_{\phi W\phi} =& -\frac{1}{2}m_H^2 g_{\mu\nu} && (5.20)\\
i\mathcal{M}_{\phi\phi\phi} =& \frac{m_H^2}{m_W^2}\left[-2x_1x_2\, k_{1\nu}k_{2\mu} + 2l_\mu l_\nu\right] && (5.21)\\
i\mathcal{M}_{WW} =& 12x_1x_2\, k_1\cdot k_2\, g_{\mu\nu} - 6l^2\, g_{\mu\nu} + 6m_W^2\, g_{\mu\nu} && (5.22)\\
i\mathcal{M}_{\phi\phi} =& \frac{m_H^2}{m_W^2}\left[2x_1x_2\, k_1\cdot k_2\, g_{\mu\nu} - l^2\, g_{\mu\nu} + g_{\mu\nu}m_W^2\right] && (5.23)\\
i\mathcal{M}_{W\phi} =& i\mathcal{M}_{\phi W} = k_1\cdot k_2\, g_{\mu\nu}\left(-x_2 + x_1x_2\right) + \frac{1}{2}m_W^2 g_{\mu\nu} - \frac{1}{2}l^2\, g_{\mu\nu} && (5.24)\\
i\mathcal{M}_{\eta^+} =& i\mathcal{M}_{\eta^-} = \frac{1}{2}x_1x_2k_{1\nu}k_{2\mu} - \frac{1}{2}l_\mu l_\nu && (5.25)
\end{aligned}$$

We remark that all matrix elements are symmetric with respect to the the swapping $(k_1 \leftrightarrow k_2, \mu \leftrightarrow \nu)$, hence we can take the crossed diagrams into account just by doubling their non-crossed counterparts.

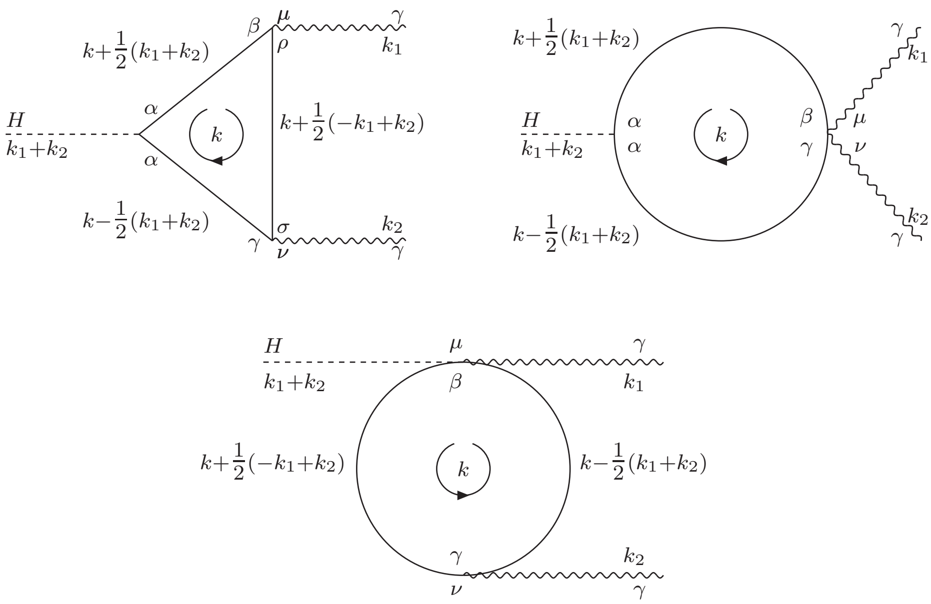


FIGURE 5.2: Loop momentum choices for the different topologies of Feynman diagrams in R_ξ gauge.

$$\begin{aligned}
i\mathcal{M}_a =& 2i\mathcal{M}_{WWW} + 2i\mathcal{M}_{\phi WW} + 2i\mathcal{M}_{WW\phi} + 2i\mathcal{M}_{W\phi W} + 2i\mathcal{M}_{W\phi\phi} \\
& + 2i\mathcal{M}_{\phi\phi W} + i\mathcal{M}_{WW} + 2i\mathcal{M}_{W\phi} + 2i\mathcal{M}_{\phi W} + 2i\mathcal{M}_{\eta^+} + 2i\mathcal{M}_{\eta^-} \\
=& 2k_{1\nu}k_{2\mu}\left(8 - 12x_1x_2\right) + 2k_1\cdot k_2\, g_{\mu\nu}\left(-7 - x_2 + x_1 + 6x_1x_2\right) \\
& + 24\left(l_\mu l_\nu - \tfrac{1}{4}l^2\, g_{\mu\nu}\right) + 6m_W^2\, g_{\mu\nu} \qquad (5.26)
\end{aligned}$$

$$\begin{aligned}
i\mathcal{M}_b =& 2i\mathcal{M}_{\phi\phi\phi} + i\mathcal{M}_{\phi\phi} \\
=& \frac{m_H^2}{m_W^2}\left[2x_1x_2\, k_1\cdot k_2\, g_{\mu\nu} - 4x_1x_2\, k_{1\nu}k_{2\mu} + g_{\mu\nu}m_W^2 + 4\left(l_\mu l_\nu - \tfrac{1}{4}l^2\, g_{\mu\nu}\right)\right] \qquad (5.27)
\end{aligned}$$

$$i\mathcal{M}_c = 2i\mathcal{M}_{\phi W\phi} = -m_H^2 g_{\mu\nu} \qquad (5.28)$$

We intend to follow Gastmans *et al.*'s recipe, so we perform the 4-D symmetric integration $l_\mu l_\nu \to \frac{1}{4}l^2 g_{\mu\nu}$. The divergences disappear, and we are able to perform the integration. Moreover, since the integral on Feynman parameters is symmetric in $x_1 \leftrightarrow x_2$, the term proportional to $x_1 - x_2$ in (5.26) identically vanishes.

$$\begin{aligned}\mathcal{M}_a =&\frac{e^2 g m_W}{(4\pi)^2}\int_{\text{simplex}} dx_1 dx_2\, \frac{2}{m_W^2 - 2x_1x_2\,(k_1\cdot k_2)}\Bigg[-k_{1\nu}k_{2\mu}\,(8-12x_1x_2)\\ &+ k_1\cdot k_2\, g_{\mu\nu}\,(7-6x_1x_2) - 3m_W^2\, g_{\mu\nu}\Bigg] \end{aligned} \tag{5.29}$$

$$\begin{aligned}\mathcal{M}_b =&\frac{e^2 g m_W}{(4\pi)^2}\frac{m_H^2}{m_W^2}\int_{\text{simplex}} dx_1 dx_2\, \frac{1}{m_W^2 - 2x_1x_2\,(k_1\cdot k_2)}\Bigg[-2x_1x_2\, k_1\cdot k_2\, g_{\mu\nu}\\ &+ 4x_1x_2\, k_{1\nu}k_{2\mu} - g_{\mu\nu}m_W^2\Bigg] \end{aligned} \tag{5.30}$$

$$\mathcal{M}_c =\frac{e^2 g m_W}{(4\pi)^2}\int_{\text{simplex}} dx_1 dx_2\, \frac{m_H^2 g_{\mu\nu}}{m_W^2 - 2x_1x_2\,(k_1\cdot k_2)} \tag{5.31}$$

In order to correctly perform the Dyson subtraction, we separate the matrix elements in three groups: the $\mathcal{M}_b$ contains only Goldstone bosons, and so in practice it is granted by scalar QED, and has to be separately renormalizable. Besides, at first sight the diagram $\mathcal{M}_c = 2\mathcal{M}_{\phi W\phi}$ does not depend on k_1 and k_2, and would be cancelled by Dyson subtraction; to avoid this, we treat it separately and use that $m_H^2 = 2k_1\cdot k_2$ on the mass shell.

$$\begin{aligned}\mathcal{M}_a \to \mathcal{M}_a - \mathcal{M}_a\Big|_{k_1=k_2=0} =&\frac{e^2 g m_W}{(4\pi)^2}\int_{\text{simplex}} dx_1 dx_2\, \frac{2}{m_W^2 - 2x_1x_2\,(k_1\cdot k_2)}\\ &\Bigg[-k_{1\nu}k_{2\mu}\,(8-12x_1x_2) + k_1\cdot k_2\, g_{\mu\nu}\,(7-12x_1x_2)\Bigg] \end{aligned} \tag{5.32}$$

$$\begin{aligned}\mathcal{M}_b \to \mathcal{M}_b - \mathcal{M}_b\Big|_{k_1=k_2=0} =&\frac{e^2 g m_W}{(4\pi)^2}\frac{m_H^2}{m_W^2}\int_{\text{simplex}} dx_1 dx_2\, \frac{1}{m_W^2 - 2x_1x_2\,(k_1\cdot k_2)}\\ &\Bigg[-4x_1x_2\, k_1\cdot k_2\, g_{\mu\nu} + 4x_1x_2\, k_{1\nu}k_{2\mu}\Bigg] \end{aligned} \tag{5.33}$$

$$\mathcal{M}_c =\frac{e^2 g m_W}{(4\pi)^2}\int_{\text{simplex}} dx_1 dx_2\, \frac{2k_1\cdot k_2 g_{\mu\nu}}{m_W^2 - 2x_1x_2\,(k_1\cdot k_2)} \tag{5.34}$$

We can sum the three terms and integrate on Feynman parameters (see Appendix A):

$$\begin{aligned}\mathcal{M} =& \frac{e^2 g m_W}{(4\pi)^2}\left[(k_1\cdot k_2)g_{\mu\nu} - k_{1\nu}k_{2\mu}\right]\int_{\text{simplex}} dx_1 dx_2\, \frac{16 - 24x_1x_2 - 4x_1x_2\frac{m_H^2}{m_W^2}}{m_W^2 - 2x_1x_2\,(k_1\cdot k_2)} \\ =& \frac{e^2 g}{(4\pi)^2 m_W}\left[(k_1\cdot k_2)g_{\mu\nu} - k_{1\nu}k_{2\mu}\right]\left(2 + 3\tau^{-1} + 3\left(2\tau^{-1} - \tau^{-2}\right)\, f(\tau)\right)\end{aligned} \tag{5.35}$$

where $\tau = \frac{m_H^2}{4m_W^2}$ and $f(\tau)$ is defined in (3.33). We got the old result (3.34).

To sum up, we calculated the $H \to \gamma\gamma$ amplitude in 't Hooft-Feynman gauge; we followed Gastmans *et al.* prescriptions to deal with divergent integrals, kept to four-dimensional integration, and used Dyson subtraction to recover the invariance on the final amplitude. Again, we have another proof that the difference between Gastmans *et al.*'s result and the renowned one is not a matter of regularization. Let us go back to the problematic integral.

5.2 Definiteness and cutoffs

We realized that the problem is in the integral

$$I_{\mu\nu} = \int d^4 l\, \frac{g_{\mu\nu} l^2 - 4 l_\mu l_\nu}{\left(l^2 - M^2 + i\epsilon\right)^3} \tag{5.36}$$

where $M^2 = m_W^2 - x_1x_2m_H^2$. In order to appreciate better the definiteness of the integral, we perform a Wick rotation and get to:

$$I_{\mu\nu} = i\int d^4 l\, \frac{\delta_{\mu\nu} l^2 - 4 l_\mu l_\nu}{\left(l^2 + M^2\right)^3} \tag{5.37}$$

Such an integral is ill-defined because it is not absolutely convergent, and so its value depends on the way we extend the domain of integration to the whole $\mathbb{R}^4$. The integral simply does not exist without a regulator (as Gastmans *et al.* naively say), and in this context the symmetric integration is meaningless. However, Shao *et al.* [9] decided to enforce Gastmans *et al.*'s argument by making the integral finite with a sharp spherical cutoff. Thus, the integrals are now meaningful and we can deal with them; moreover, a spherical cutoff is obviously invariant for Euclidean rotations, hence the 4-D symmetric integration works correctly. Besides, Shao *et al.* extend the use of the cutoff regularization to all one-loop integrals, modifying the Passarino-Veltman reduction method [33,34]. We left in Appendix B the main feature of the cutoff-regularized one-loop integrals, and here we only discuss the results for $H \to \gamma\gamma$. We only remark the Passarino-Veltman scheme adopts a different choice for the loop momentum, as in Fig. 5.3.

In ‘t Hooft-Feynman gauge, they get:

$$\begin{aligned}
\mathcal{M} =& \frac{e^2 g}{(4\pi)^2 m_H^2 m_W}\Big[(k_1\cdot k_2)g_{\mu\nu}\left(m_H^2+6m_W^2-12m_W^2\left(m_H^2-2m_W^2\right)C_0\right) \\
&- k_{1\nu}k_{2\mu}\left(2m_H^2+12m_W^2-12m_W^2\left(m_H^2-2m_W^2\right)C_0\right)\Big] \\
=& \frac{e^2 g}{(4\pi)^2 m_W}\left[(k_1\cdot k_2)g_{\mu\nu}\left(1+\frac{3}{2}\tau^{-1}+3\tau^{-1}\left(2-\tau^{-1}\right)f(\tau)\right)\right. \\
&\left. - k_{1\nu}k_{2\mu}\left(2+3\tau^{-1}+3\tau^{-1}\left(2-\tau^{-1}\right)f(\tau)\right)\right]
\end{aligned} \tag{5.38}$$

where $C_0 = C_0\left(0,0,m_H^2,m_W^2,m_W^2,m_W^2\right) = -2f(\tau)/m_H^2$ is the 3-point scalar function, and as usual $\tau = m_H^2/4m_W^2$. As we expect, gauge invariance is spoiled with a cutoff regularization, and we must perform the Dyson subtraction. On the mass shell, we have $k_1\cdot k_2 = \frac{1}{2}m_H^2 = 2m_W^2\tau$, and so we can evaluate the matrix element in $k_1 = k_2 = 0$ by taking the limit $\tau\to 0$. Since $f(\tau) \xrightarrow{\tau\to 0} \tau\left(1+\frac{1}{3}\tau\right)+O\left(\tau^3\right)$, we have:

$$\begin{aligned}
\mathcal{M}\Big|_{k_1=k_2=0} =& \frac{e^2 g}{(4\pi)^2 m_W}\left[\left(2m_W^2\tau\right)g_{\mu\nu}\left(1+\frac{3}{2}\tau^{-1}+3\tau^{-1}\left(2-\tau^{-1}\right)\tau\left(1+\frac{1}{3}\tau\right)\right)\right] \\
=& \frac{e^2 g}{(4\pi)^2 m_W}\left(2m_W^2\tau\right)g_{\mu\nu}\left(\frac{3}{2}\tau^{-1}-3\tau^{-1}+O(1)\right) \\
\to& \frac{e^2 g m_W}{(4\pi)^2}\left(-3g_{\mu\nu}\right)
\end{aligned} \tag{5.39}$$

Moreover, as we discussed before, by subtracting $\mathcal{M}\Big|_{k_1=k_2=0}$ we are taking off the contribution of the $\mathcal{M}_{\phi W\phi}$ diagram, which does not depend on k_1, k_2. We can perform the subtraction and re-add the diagram:

$$\mathcal{M}_{\phi W\phi} = \frac{e^2 g}{(4\pi)^2 m_W}\left(\frac{1}{2}m_H^2 g_{\mu\nu}\right) \tag{5.40}$$

and get to

$$
\begin{aligned}
\mathcal{M} \to & \mathcal{M} - \mathcal{M}\Big|_{k_1=k_2=0} + \mathcal{M}_{\phi W \phi} = \frac{e^2 g}{(4\pi)^2 m_W}\Bigg[\\
& (k_1 \cdot k_2) g_{\mu\nu} \left(1 + \frac{3}{2}\tau^{-1} + 3\tau^{-1}\left(2-\tau^{-1}\right) f(\tau)\right) \\
& - k_{1\nu} k_{2\mu} \left(2 + 3\tau^{-1} + 3\tau^{-1}\left(2-\tau^{-1}\right) f(\tau)\right) \\
& + 3 m_W^2 g_{\mu\nu} \frac{2k_1 \cdot k_2}{m_H^2} + \frac{1}{2} m_H^2 g_{\mu\nu} \frac{2k_1 \cdot k_2}{m_H^2}\Bigg] \\
= & \frac{e^2 g}{(4\pi)^2 m_W}\left[(k_1 \cdot k_2) g_{\mu\nu} - k_{1\nu} k_{2\mu}\right]\left(2 + 3\tau^{-1} + 3\tau^{-1}\left(2-\tau^{-1}\right) f(\tau)\right)
\end{aligned}
\tag{5.41}
$$

In this way, we see that the Passarino-Veltman reduction with a cutoff reproduces the calculations of the last section in ‘t Hooft-Feynman gauge, recovering the old result (3.34). We wonder whether the same happens in unitary gauge. Indeed, we have

$$
\begin{aligned}
\mathcal{M} = & \frac{e^2 g}{(4\pi)^2 m_H^2 m_W}\Big[(k_1 \cdot k_2) g_{\mu\nu}\left(6m_W^2 - 12 m_W^2\left(m_H^2 - 2m_W^2\right) C_0\right) \\
& - k_{1\nu} k_{2\mu}\left(12 m_W^2 - 12 m_W^2 \left(m_H^2 - 2m_W^2\right) C_0\right)\Big] \\
= & \frac{3e^2 g}{(4\pi)^2 m_W}\Bigg[(k_1 \cdot k_2) g_{\mu\nu}\left(\frac{1}{2}\tau^{-1} + \tau^{-1}\left(2-\tau^{-1}\right) f(\tau)\right) \\
& - k_{1\nu} k_{2\mu}\left(\tau^{-1} + \tau^{-1}\left(2-\tau^{-1}\right) f(\tau)\right)\Bigg]
\end{aligned}
\tag{5.42}
$$

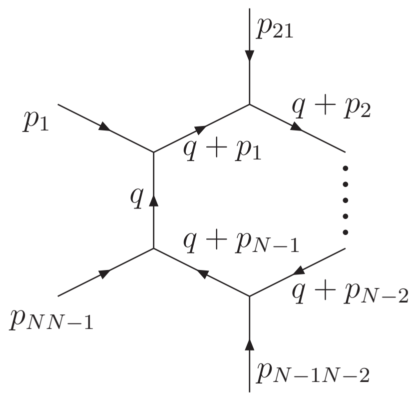


Figure 5.3: Conventions for the N-point integral.

Yet, we evaluate $\mathcal{M}\Big|_{k_1=k_2=0} = \frac{-3e^2 g m_W}{(4\pi)^2} g_{\mu\nu}$ and subtract:

$$\begin{aligned}
\mathcal{M} \to & \mathcal{M} - \mathcal{M}\Big|_{k_1=k_2=0} = \frac{3e^2 g}{(4\pi)^2 m_W}\Bigg[\\
& (k_1 \cdot k_2) g_{\mu\nu} \left(\frac{1}{2}\tau^{-1} + \tau^{-1}\left(2 - \tau^{-1}\right) f(\tau)\right) \\
& - k_{1\nu} k_{2\mu} \left(\tau^{-1} + \tau^{-1}\left(2 - \tau^{-1}\right) f(\tau)\right) \\
& + m_W^2 g_{\mu\nu} \frac{2k_1 \cdot k_2}{m_H^2}\Bigg] \\
= & \frac{3e^2 g}{(4\pi)^2 m_W} \left[(k_1 \cdot k_2) g_{\mu\nu} - k_{1\nu} k_{2\mu}\right] \left(\tau^{-1} + \tau^{-1}\left(2 - \tau^{-1}\right) f(\tau)\right)
\end{aligned} \tag{5.43}$$

This is interesting: Shao *et al.* enforce the Gastmans *et al.*'s argument by properly choosing a regulator which allows them to perform the 4-D symmetric integration; the final Dyson subtraction leads to Gastmans *et al.*'s result (3.32) in unitary gauge, whereas the old result (3.34) is found again in ‘t Hooft-Feynman gauge.

Moreover, Shao *et al.* provide an estimate of potential surface term which could appear with different choices of loop momentum because of highly divergent integrals in unitary gauge. If we shift the Passarino-Veltman loop momentum by p, we get an additional surface term:

$$\begin{aligned}
\Delta\mathcal{M}_{\mu\nu}(p) = & \frac{e^2 g}{6 (4\pi)^2 m_W^3} \\
& \times \Big[2 g_{\mu\nu} (k_1 - k_2)\cdot p \left(-3\Lambda^2 - 2m_H^2 + 3m_W^2 + (k_1 + k_2)\cdot p - p^2\right) \\
& + (k_{2\mu} p_\nu - p_\mu k_{1\nu}) \left(-3\Lambda^2 - 2m_H^2 - 6m_W^2 + (k_1 + k_2)\cdot p - p^2\right)\Big]
\end{aligned} \tag{5.44}$$

If we choose $p = (k_1 + k_2)/2$ as Gastmans *et al.* do, we get $\Delta\mathcal{M} = 0$. There is no hope of adding surface terms to make Gastmans *et al.*'s result agree with the old one.

5.3 Other R_ξ gauges

The main difference between renormalizable R_ξ gauges and the unitary gauge is in the ultraviolet behaviour of the propagator

$$\begin{aligned} i\Delta^{\mu\nu} =& \frac{1}{q^2 - m_W^2}\left(g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2 - \xi m_W^2}\right) \\ =& \frac{g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2}}{q^2 - m_W^2} + \frac{q^\mu q^\nu}{m_W^2}\frac{1}{q^2 - \xi m_W^2} \end{aligned} \tag{5.45}$$

For finite ξ, the propagator is $O\left(q^{-2}\right)$, and leads to the same logarithmic divergent integral as in ‘t Hooft-Feynman gauge; on the other hand, for $\xi = \infty$ (unitary gauge) the propagator is $O\left(q^0\right)$, leading to a highly-divergent amplitude.

In dimensional regularization we can freely shift even the highly divergent integrals; hence we could show that all ξ-dependent terms in the amplitude cancel out, leading to the same result as in the unitary gauge [7]. We provide the explicit calculations of this issue in Appendix C.

On the contrary, in gauge-breaking regularizations, highly divergent integrals cannot be freely shifted, and it is difficult to show the explicit cancellation. We limit ourselves to studying the divergent behaviour in R_ξ gauges. The integrals are only logarithmic divergent, so we can perform usual Feynman parametrization and shift the integration variables. We obtain the expression

$$\begin{aligned} i\mathcal{M}^{\mu\nu}_{\xi<\infty} =& \frac{e^2 g}{(2\pi)^4 m_W}\left(6m_W^2 + m_H^2\right) 5! \int_0^1 \prod_{i=1}^{6} dx_i\, \delta\left(1 - \sum_{i=1}^{6} x_i\right) \\ &\cdot \int d^4 l\, \frac{l^6}{\left(l^2 - M^2 + i\epsilon\right)^6}\left(4l^\mu l^\nu - l^2 g^{\mu\nu}\right) + \text{finite integrals} \end{aligned} \tag{5.46}$$

with $M^2 = m_W^2\left(1 + (\xi - 1)\left(x_4 + x_5 + x_6\right)\right) - m_H^2\left(x_3 + x_6\right)\left(x_1 + x_4\right)$.

Despite of the complication of Feynman parameters, we have the same form as in (6.22): the divergent integral can be regularized leading to a tensor $Ig^{\mu\nu}$, whereas the coefficient of $k_2^\mu k_1^\nu$ is finite and uniquely determined. If we use a gauge-invariant regularization, the coefficient of $g^{\mu\nu}$ will be the same as the one of $k_2^\mu k_1^\nu$, so that the amplitude will be properly transverse. On the other hand, with a cutoff we have in general a different value for the coefficient of $g^{\mu\nu}$, but it will be tuned to be the same as the one of $k_2^\mu k_1^\nu$ by Dyson subtraction.

Chapter 6

Indeterminate integrals

In the previous chapters we showed that if we use a renormalizable gauge with a cutoff, or if we use dimensional regularization in whatever gauge, we get to the old result (3.34) for the amplitude $H \to \gamma\gamma$. If we instead use a cutoff with unitary gauge, we get to the Gastmans *et al.*'s one. Our aim is now to understand how this difference arises, and if we can still trust the good old cutoff.

Let us go back again to the integral

$$I_{\mu\nu} = \int d^4l \, \frac{g_{\mu\nu} l^2 - 4 l_\mu l_\nu}{\left(l^2 - M^2 + i\epsilon\right)^3} \tag{6.1}$$

where $M^2 = m_W^2 - x_1 x_2 m_H^2$. After Wick rotation [1] and rescaling $l \to Ml$, we get

$$I_{\mu\nu} = i \int d_4 l \, \frac{\delta_{\mu\nu} l^2 - 4 l_\mu l_\nu}{\left(l^2 + 1\right)^3} \tag{6.2}$$

To simplify the discussion, let us focus on the case $\mu = 1, \nu = 1$

$$I_{11} = i \int d_4 l \, \frac{l^2 - 4 l_1^2}{\left(l^2 + 1\right)^3} = i \int d_4 l \, F_{11}\left(l\right) \tag{6.3}$$

The integrand is not a summable function: it is not positive everywhere in the domain of integration, and $\int d_4 l \, |F_{11}| = \infty$, which means that the integral is not defined *per se* – the value depends on how the boundary is chosen to behave at infinity.

[1] Which amounts to $d^4 l = i d_4 l$, $g_{\mu\nu} \to \delta_{\mu\nu}$, $l^2 \to -l_E^2$ and $l_\mu l_\nu \to -l_\mu^E l_\nu^E$.

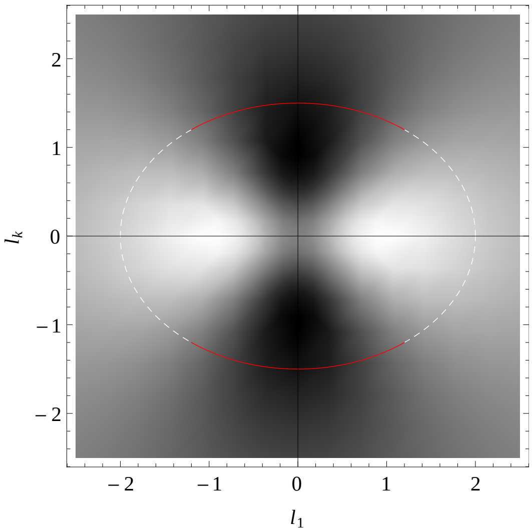


FIGURE 6.1: We integrate $F_{11} = \frac{l^2-4l_1^2}{(1+l^2)^3}$ over an elliptic domain. Because of cylindrical symmetry, the plot is the same independently of $l_k = l_2, l_3, l_4$. The darker the background, the larger is the F_{11} value. The boundary is solid when $F_{11} > 0$, dashed otherwise. Since we broke the spherical symmetry, the ellipse bounds a negative part which is larger than the positive one: the integral does not vanish.

As a first example, let us consider a 'spherical cutoff' in the sense described below. In polar coordinates, we write

$$\begin{aligned} I_{11} &= i \int_0^\Lambda dl \, \frac{l^5}{(1+l^2)^3} \int d\Omega_4 \, \left(1 - 4\cos^2\theta\right) \\ &= i4\pi \int_0^\Lambda dl \, \frac{l^5}{(1+l^2)^3} \int_0^\pi d\theta \, \sin^2\theta \left(1 - 4\cos^2\theta\right) = 0 \end{aligned} \tag{6.4}$$

Λ is a dimensionless cutoff, being l a dimensionless integration variable. The angular part vanishes, so there are no problems with the logarithmic divergence of the radial part. Actually, every integration domain which has the $l_i \leftrightarrow \pm l_j$ symmetry, leads to an identically vanishing integral.

As a second case we choose a non-symmetrical domain of integration. For example, let us integrate F_{11} over the elliptical domain $\frac{l_1^2}{1+\epsilon} + l_2^2 + l_3^2 + l_4^2 \leq \Lambda^2$ (see Figure 6.1)

$$\begin{aligned} I_{11} &= i4\pi\sqrt{1+\epsilon} \int_0^\Lambda dl \, l^5 \int d\theta \sin^2\theta \, \frac{1-(4+3\epsilon)\cos^2\theta}{\left(1+l^2+l^2\epsilon\cos^2\theta\right)^3} \\ &\xrightarrow{\Lambda\to\infty} i\pi^2 \frac{8+4\epsilon-\epsilon^2-8\sqrt{1+\epsilon}}{2\epsilon^2} \end{aligned} \tag{6.5}$$

The integral in (6.5) can assume different finite values as a function of ϵ.

Choosing asymmetric boundaries, we even lose tensor invariance, obtaining a 4×4 matrix of unrelated, indeterminate terms. This translates into the fact that $I_{\mu\nu}$ is no longer proportional to $\delta_{\mu\nu}$ as it should be the case (see (6.2)). We seek an appropriate choice of the boundaries for all the terms in the $I_{\mu\nu}$ matrix in such a way as to recover a $\delta_{\mu\nu}$ structure. We can therefore compute the $I_{\mu\nu}$ entries by choosing the same asymmetric boundary on all diagonal terms, and generic symmetric boundaries for all off-diagonal terms. In this way, all diagonal terms will have the same indeterminate value I, whereas off-diagonal terms will vanish. We thus obtain $I_{\mu\nu} = I\delta_{\mu\nu}$, with I being an indeterminate (even divergent) constant. This trick may seem wooly, but it is only a recipe to recover *a posteriori* a correct tensor form, just as we did with Dyson subtraction to recover gauge invariance.

For the sake of simplicity in the following we consider a two-dimensional version of $I_{\mu\nu}$ in (6.2)

$$I_{\mu\nu} = i\int d_2l\,\frac{\delta_{\mu\nu}l^2 - 2l_\mu l_\nu}{\left(l^2+1\right)^2} \tag{6.6}$$

This version of $I_{\mu\nu}$ has the same properties of its four-dimensional counterpart, namely: *i*) the integral is superficially divergent as a logarithm, *ii*) it is identically zero for symmetric integration domains, *iii*) the integrand function has no definite sign. The conclusions we will draw from the following calculations in two dimensions remain unaltered in four dimensions: here we just avoided superfluous technical complications.

As we did before, let us start by computing the I_{11} term. We realize that I_{11} can be mapped into an entry of the I_{12} kind upon a rotation by 45^o of l_1l_2 axes. $I_{11} = I_{12}$ only if the integration domain is rotated accordingly. Since the calculations turn out to be simpler using the $\{12\}$ entry, we will make our observations on this case only

$$I_{12} = i\int d_2l\frac{-2l_1l_2}{\left(1+l^2\right)^2} = i\int d_2l\,F_{12} \tag{6.7}$$

At any rate we remark that the domains of integrations will be chosen in such a way that eventually all off-diagonal $I_{\mu\nu}$ entries will vanish so as to recover eventually the $\delta_{\mu\nu}$ tensor structure.

The integrand in (6.7) is negative when $l_1l_2 > 0$ (I and III quadrant), and positive otherwise. In the former case, we bound a domain with two quarters of a circumference of radius Λ; in the latter case we use a square with edge Λ (Figure 6.2).

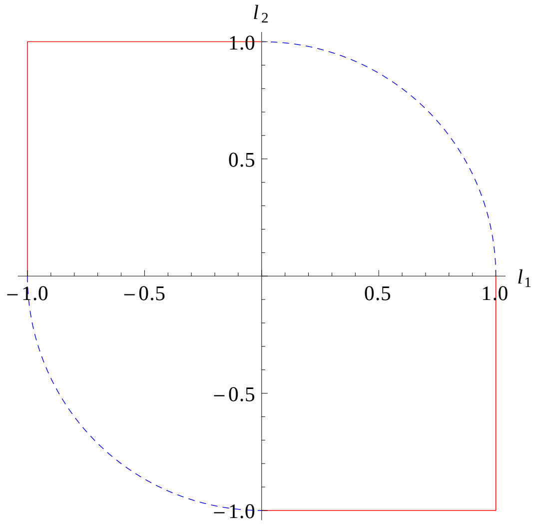


FIGURE 6.2: We integrate $F_{12} = \frac{-2l_1 l_2}{(1+l^2)^2}$ over a mixed boundary. We bound the domain with a circle when $F_{12} < 0$, and with a square otherwise. Since we broke spherical symmetry, the integral does not vanish.

We have

$$\begin{aligned} I_{12} &= -4i \int_0^{\Lambda} dl \, \frac{l^3}{(1+l^2)^2} \int_0^{\pi/2} d\theta \sin\theta \cos\theta + 4i \int_{[0,\Lambda]\times[0,\Lambda]} dl_1 dl_2 \, \frac{l_1\, l_2}{(1+l_1^2+l_2^2)^2} \\ &= i\left(\frac{\Lambda^2}{1+\Lambda^2} + \ln\frac{1+\Lambda^2}{1+2\Lambda^2}\right) \xrightarrow{\Lambda\to\infty} i\left(1+\ln\frac{1}{2}\right) \end{aligned} \tag{6.8}$$

Again we get a finite non-zero value. The leading divergences are the same in each quadrant, whereas the finite part is boundary-dependent, so that the sum does not vanish.

More generally, we can slice $\mathbb{R}^2$ into a countable set of bounded regions, in order to reduce the integral over the whole $\mathbb{R}^2$ to a countable sum of finite integrals, i.e., into a series. We can thus use the Riemann rearrangement theorem [35] to obtain whatever finite value or logarithmic divergence.

For example, let us consider all the concentric circumferences with integer radius, thus slicing $\mathbb{R}^2$ into annuli: the integral of F_{12} over each annulus vanishes by circular symmetry. Therefore we slice each annulus into a positive region P_k where

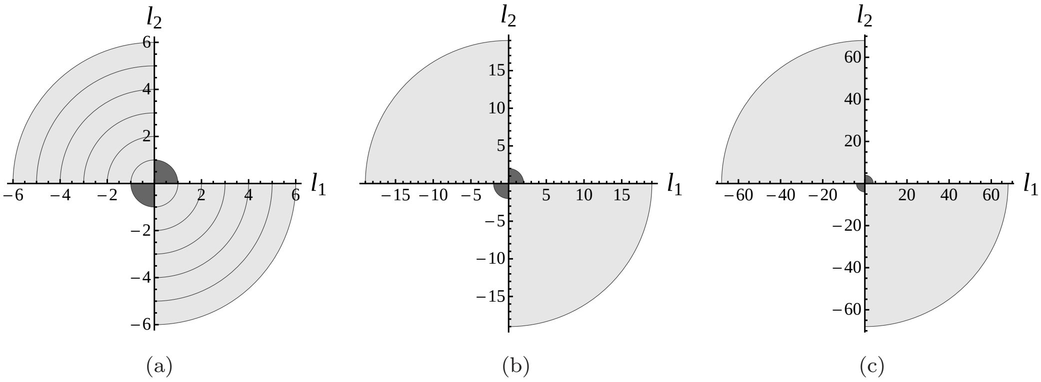


FIGURE 6.3: Riemann rearrangement. Light gray regions have positive integral p_k, dark gray regions have negative integrals n_k. The absolute value of each region is bounded by $M \approx 0.62$. We can obtain a divergent sum following this algorithm: (a) we sum first positive terms $p_0 + \cdots + p_{N_1}$, until we exceed $1 + M$, then we can subtract n_0 still exceeding 1; (b) we continue adding positive terms until we exceed $2 + M$, then we can subtract n_1 still exceeding 2; (c) and so on. We see that the negative region becomes smaller and smaller than the positive region, so that it cannot cancel the logarithmic divergence.

$F_{12} > 0$, and a negative region N_k where $F_{12} < 0$ (Figure 6.3). We therefore have

$$\begin{aligned} p_k &= \int_{P_k} d_2 l\, F_{12} = 4 \int_0^{\pi/2} d\theta \sin\theta \cos\theta \int_k^{k+1} dl\, \frac{l^3}{(1+l^2)^2} \\ &= \frac{1}{k^2+2k+2} - \frac{1}{k^2+1} + \log \frac{k^2+2k+2}{k^2+1} \\ n_k &= \int_{N_k} d_2 l\, F_{12} = -p_k \end{aligned} \tag{6.9}$$

The p_k form a bounded sequence of positive terms converging to 0. We can find that the greatest term of the sequence is $p_1 = M \approx 0.62$. Specularly, the n_k form a sequence of negative terms converging to zero, bounded by $n_1 = -M \approx -0.62$. If we unite all P_k and N_k, we recover the whole $\mathbb{R}^2$, therefore if we sum all p_k and n_k we recover the whole integral. Since $\sum_k p_k$ and $\sum_k n_k$ both diverge separately, we must specify the correct ordering of terms. We start by adding the first positive terms p_k until we exceed $1 + M$, and then add the first negative term n_0. Since all $0 > n_k > -M$, we still have

$$p_0 + p_1 + \cdots + p_{N_1} - |n_0| > 1 \tag{6.10}$$

We can continue adding positive terms until we exceed $2 + M$, and then add n_1, and so on. The resulting sum covers all P_k and N_k regions. The series diverges, and so does the integral.

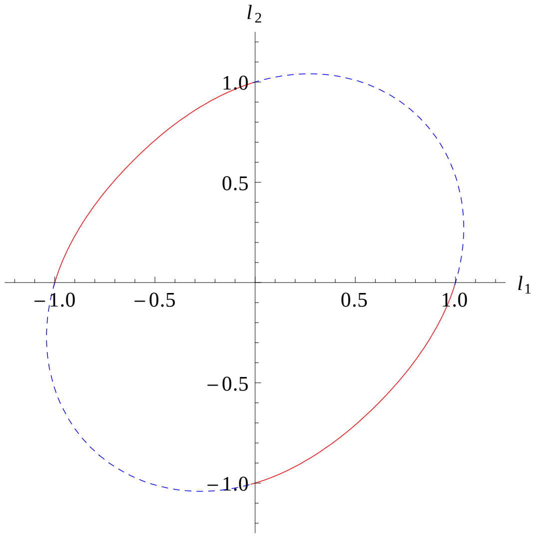


FIGURE 6.4: We integrate $F_{12} = \frac{-2l_1 l_2}{(1+l^2)^2}$ regulating the function with a smooth cutoff Λ. In the plot we show a deformed cutoff $\Lambda(\theta) = \Lambda e^{\epsilon \sin 2\theta}$. The curve is solid in the region where $F_{12} > 0$, dashed otherwise. Since we broke spherical symmetry, the integral does not vanish.

One might wonder whether the same behaviour occurs with a smooth cutoff. We calculate (6.7) with a Schwinger regulator [36]

$$\begin{aligned} I_{12} &= i \int d_2 l \, (-2l_1 l_2) \int_{\frac{1}{\Lambda^2}}^{\infty} ds \, s \, e^{-s\left(1+l^2\right)} \\ &= -\, i \, \Gamma\left(0, \frac{1}{\Lambda^2}\right) \int_0^{2\pi} d\theta \, \sin\theta \cos\theta = 0 \end{aligned} \tag{6.11}$$

where $\Gamma(a,b)$ is the incomplete Gamma function [37]. Again, the angular part of the integral vanishes, and so we can ignore the logarithmic divergence in the radial part. However, if we deform the cutoff giving an angular dependency to it, e.g. $\Lambda \to \Lambda(\theta) = \Lambda \exp(\epsilon \sin 2\theta)$ (Figure 6.4), we obtain

$$\begin{aligned} I_{12} &= i \int d_2 l \, (-2l_1 l_2) \int_{\frac{1}{\Lambda^2(\theta)}}^{\infty} ds \, s \, e^{-s\left(1+l^2\right)} \\ &= -\, i \int_0^{2\pi} d\theta \, \sin\theta \cos\theta \, \Gamma\left(0, \frac{1}{\Lambda^2(\theta)}\right) \xrightarrow{\Lambda\to\infty} -i\pi\epsilon \end{aligned} \tag{6.12}$$

Schwinger regularization turns out to be particularly instructive as the indeterminacy of the integrals is not related to the shape of the integration domain but rather to the behaviour at infinity of the integrand function. This is a sign of the fact that, in order to tackle the indeterminacy of the critical integrals in this calculations, one exclusively needs to set *physical constraints* as there is no mathematical prescription which can univocally determine them.

In fact, one might ask if gauge symmetry could be just such a physical prescription. If so, all gauge-invariant regularizations would lead to the same univocal value. We can indeed show this feature with an argument by Shifman *et al.* [6].

To pinpoint this issue, we rewrite the integral (6.1) in the form

$$I_{\mu\nu} = \frac{1}{2}\int d^4l \frac{\partial^2}{\partial l^\mu \partial l^\nu}\frac{1}{l^2 - M^2 + i\epsilon} - g_{\mu\nu}\int d^4l\,\frac{M^2}{(l^2 - M^2 + i\epsilon)^3} \tag{6.13}$$

The second integral is well defined and gives just a value $-i\pi^2/2$. The first integral of total derivatives which cancels this number in dimension 4 is the one which breaks gauge invariance. Indeed, it can be viewed as a second order of expansion in the constant gauge potential A_μ of the expression

$$\int d^4l\,\frac{1}{(l + eA)^2 - M^2 + i\epsilon}\,. \tag{6.14}$$

Thus, the latter integral must vanish in all regularizations which preserve gauge invariance. Hence, the original integral (6.1) has a univocal value of $-ig_{\mu\nu}\pi^2/2$ in all gauge-invariant regularizations.

To summarize: in the $H \to \gamma\gamma$ amplitude, we deal with an ill-defined integral. If we use gauge invariant regulators, the integral has a univocal value, and the calculations lead to the renowned result (3.34). On the other hand, if we decide to use gauge breaking regulators, we must remember that the value of the integral depends on the choice of the regularization scheme, and we cannot rely on the fact that it seems finite and univocal. In other words, the value of the integral must be considered indeterminate anyway.

Does this fact mean that a gauge breaking scheme like the cutoff is not to be pursued at all? In the following we mean to show that the problem does not reside in the use of a cutoff scheme itself but rather in figuring out that different ways (spherical, elliptical, etc.) of implementing a cutoff scheme amount to different values of integrals, *i.e.*, to indeterminate coefficients. We can actually use some cutoff schemes provided that there is a clear recipe on how to absorb the indeterminate coefficients arising in the calculation, also restoring gauge invariance. *We will show that such a recipe cannot be found in unitary gauge.*

6.1 A QED example

After this discussion one might therefore ask why cutoff renormalization works in several cases independently of the issue of the indeterminate constants discussed above.

For example, let us discuss the cutoff renormalization of vacuum polarization in QED. Since we are not protected by Ward identity, the integral presents a quadratic divergence

$$\Pi_2^{\mu\nu}(k) = -4ie_0^2 \int^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{2p^\mu p^\nu - g^{\mu\nu}p^2}{(p^2 - m^2)\left((p-k)^2 - m^2\right)} \propto e_0^2 \Lambda^2 g^{\mu\nu} \tag{6.15}$$

The integral is divergent and sign-undefined; it is ill-defined as the one in (6.1). We could repeat the above considerations to show that we can lower the degree of divergence with an appropriate choice of boundary. Anyway we simply add a photon mass counterterm and force the quadratic term to vanish (i.e. to have a massless photon), in order to recover a good gauge invariant theory. The presence of the counterterm ensures that the quadratic divergence disappears *whatever value the integral has.* In this case, we can forget about the indeterminacy itself, and calculate the integral with the boundary we prefer. On the other hand, we cannot choose a boundary and claim that an integral like the one in (6.1) has a finite value and needs no counterterm (as Gastmans *et al.* did); we must consider expressly the indeterminacy and add a counterterm to absorb it.

6.2 Finite but indeterminate

In a beautiful paper, Jackiw [38] gave a different point of view of such indeterminacy, showing that it can occur if the regulator has not the full symmetry of the theory. For example, he took the vacuum polarization in Schwinger model [39] (namely a 2-D QED):

$$\Pi^{\mu\nu}(p) = i \,\mathrm{tr} \int \frac{d^2k}{(2\pi)^2} \gamma^\mu \frac{i}{\not{k}} \gamma^\nu \frac{i}{\not{k} + \not{p}} \tag{6.16}$$

This integral is logarithmically divergent, therefore we can shift the integration variable, and then separate a divergent and a convergent part:

$$\Pi^{\mu\nu}(p) = \mathrm{tr}\, \tfrac{1}{2} \gamma^\mu \gamma_\alpha \Pi^{\alpha\nu}_\infty + \tfrac{1}{\pi} \left(\frac{g^{\mu\nu}}{2} - \frac{p^\mu p^\nu}{p^2} \right) \tag{6.17}$$

where

$$\Pi^{\mu\nu}_\infty \equiv 2i \int \frac{d^2k}{(2\pi)^2} \frac{(-k^2 g^{\mu\nu} + 2k^\mu k^\nu)}{(k^2 - \mu^2)^2} \tag{6.18}$$

and μ^2 is an arbitrary infrared cutoff, whose value does not affect $\Pi^{\mu\nu}_\infty$[2]. We remark that $\Pi^{\mu\nu}$ is the 2-D version of Gastmans *et al.*'s amplitude (3.28) before the symmetric integration and Dyson subtraction. In fact, both $\Pi^{\mu\nu}(p)$ and (3.28) seem not to be gauge invariant – they are not transverse to external momenta; moreover, the divergent part $\Pi^{\mu\nu}_\infty$ is just the 2-D version of the integral (6.7); finally, the tensor $\Pi^{\mu\nu}$ is traceless in (μ, ν), as we can see in (6.16) when it is remembered that in two dimensions $\gamma^\mu \not{k} \gamma_\mu = 0$.

To make progress we must assign a value to $\Pi^{\mu\nu}_\infty$. But no unique value can be given, because the integral is divergent, that is, undefined. By Lorentz invariance, $\Pi^{\mu\nu}_\infty$ should be proportional to $g^{\mu\nu}$. In two dimensions any Lorentz-invariant prescription for calculating the integral will give a vanishing value, $\Pi^{\mu\nu}_\infty = 0$, consistent with its being proportional to $g^{\mu\nu}$ and traceless.

Otherwise, we can choose a gauge-invariant regulator like dimensional regularization, which gives an additional contribution $\Pi^{\mu\nu}_\infty = \frac{1}{2\pi} g^{\mu\nu}$, and leads to a gauge-invariant result for $\Pi^{\mu\nu}(p)$.

Thus, we have just shown that different choices for the regularization scheme can produce different results; therefore, Jackiw proposes the *Ansatz* that $\Pi^{\mu\nu}_\infty = a g^{\mu\nu}$, where a is dimensionless and as yet *indeterminate*. In this viewpoint, the Feynman graphs of the Schwinger model need not be regulated, but give a vacuum polarization with an *indeterminate local part*:

$$\Pi^{\mu\nu}(p) = \tfrac{1}{\pi} \left(g^{\mu\nu} \left(\frac{1+a}{2} \right) - \frac{p^\mu p^\nu}{p^2} \right) \tag{6.19}$$

[2]After the Wick rotation, μ can be absorbed by substituting $k \to \mu k$

This result is still unpleasant, because of the ambiguity of a and the lack of transversality. Yet, we can still make use of the formal gauge invariance of the Schwinger model and force *a posteriori* gauge symmetry to be preserved. Indeed this constraint yields to the univocal choice $a = 1$ and to the conventional result for the vacuum polarization in this model

$$\Pi^{\mu\nu}(p) = \tfrac{1}{\pi}\left(g^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2}\right) \tag{6.20}$$

and a photon mass

$$m^2 = \frac{e^2}{\pi} \tag{6.21}$$

We remark that by adopting the transverse expression for the vacuum polarization, in order to agree with the constraint of gauge invariance, we are compelled to abandon the tracelessness of $\Pi^{\mu\nu}(p)$.

Let us go back to $H \to \gamma\gamma$. The argument by Jackiw has shown that indeterminacy can arise if we use regulators which have less symmetry than the theory, and it is not necessarily resolved when we restore the symmetry at the end of the calculation. On the other hand, if the regulator maintains the symmetry, the result will be univocal, as shown in the last section. We can easily understand the case of gauge symmetry: gauge invariant regulators decrease the degree of divergence of the integral, making it finite and regulator independent. Indeed, by naive power counting, we know that the amplitude $H \to \gamma\gamma$ is logarithmically divergent in renormalizable gauges. In gauge invariant regularizations, we can group two momentum powers in the numerator to extract the gauge invariant factor $k_1 \cdot k_2\, g^{\mu\nu} - k_2^{\mu}k_1^{\nu}$, so that the amplitude becomes finite. In unitary gauge too, we expect the same finite amplitude after a non-straightforward cancellation of higher divergent terms. However, this cannot be done in cutoff regularization where the Ward identity and gauge invariance are spoiled by the breaking of shift invariance. In the latter case the integral remains indeterminate or at worst logarithmically divergent. The expectations by Gastmans *et al.* to get a finite amplitude which needs no regulator are disappointed by the choice of four-dimensional symmetric integration, which implicitly uses a spherical cutoff scheme, leading to the breaking of gauge invariance and to a divergent amplitude.

To subtract the divergence in the cutoff regularization scheme and, in general, all cutoff-dependent terms, we need some counterterms. Breaking gauge symmetry, we have the most general lagrangian with all possible combinations of bare fields and bare couplings, even an *ad hoc* counterterm in the form $\delta m_{0A}\, h_0 A_0^2$. This is what Dyson subtraction means: hide all divergent, cutoff-dependent, non-gauge-invariant terms into a counterterm, which would vanish in a gauge invariant regularization scheme.

We recall here the result of Sec. 5.1 about the $H \to \gamma\gamma$ amplitude in 't Hooft-Feynman gauge. Before regularizing and calculating divergent integrals, we found:

$$\begin{aligned}\mathcal{M}^{\mu\nu}_{\xi=1} =& \frac{e^2 g}{(4\pi)^2 m_W}\Bigg[-k_2^\mu k_1^\nu \left(2+3\tau^{-1}+3\tau^{-1}\left(2-\tau^{-1}\right)f(\tau)\right) \\ & - 2m_H^2\left(1+\frac{3}{2}\tau^{-1}\right)\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^4 l}{i\pi^2}\frac{g^{\mu\nu} l^2 - 4 l^\mu l^\nu}{\left(l^2 - 1 + 4x_1 x_2 \tau + i\epsilon\right)^3} \\ & + \frac{1}{2} m_H^2 g^{\mu\nu}\left(1+\frac{3}{2}\tau^{-1}+3\tau^{-1}\left(2-\tau^{-1}\right)f(\tau)\right)\Bigg]\end{aligned} \tag{6.22}$$

We remark that the first term (proportional to $k_2^\mu k_1^\nu$) contains only well-defined finite integrals; the second term is to be considered indeterminate (vanishing according to symmetric integration as in Gastmans *et al.*). With the use of DREG, the second term would give $\frac{1}{2}m_H^2 g^{\mu\nu}\left(1+\frac{3}{2}\tau^{-1}\right)$, leading to the well known gauge-invariant expression. However, let us remain in the framework of gauge-breaking regularizations.

In Sec. 5.1, a modified version of Dyson subtraction is performed to recover gauge invariance and get to the final expression of the amplitude. One might even wonder whether Dyson subtraction is allowed without divergent terms [8]. As we have just shown, the integral in the second term is probably divergent and in any case cutoff-dependent, so we are allowed to add a counterterm and impose gauge invariance as a renormalization condition. In so doing we get the correct amplitude in (3.34). We would have the same expression by using symmetric integration: every value of the integral disappears into the counterterm. We are therefore led to observe that the arbitrariness related to the choice of the boundary (or, in general, of the regulator) is solved by imposing gauge invariance.

Why is Gastmans *et al.*'s amplitude different from the standard one? As shown in (3.26), in unitary gauge we have another divergent integral

$$\begin{aligned} F =& \frac{2}{m_W^2} \int d^4 l \int_{\text{simplex}} dx_1 dx_2 \frac{1}{\left(l^2 - m_W^2 + 2x_1 x_2 \left(k_1 \cdot k_2\right)\right)^3} \\ &\times \Big[4 l^\alpha l^\beta \left(g_{\mu\alpha} g_{\nu\beta} \left(k_1 \cdot k_2\right) - g_{\mu\alpha} k_{1\nu} k_{2\beta} - g_{\nu\beta} k_{1\alpha} k_{2\mu} + g_{\mu\nu} k_{1\alpha} k_{2\beta}\right) \\ &+ 2 l^2 \left(k_{1\nu} k_{2\mu} - g_{\mu\nu} \left(k_1 \cdot k_2\right)\right) \Big] \end{aligned} \tag{6.23}$$

By symmetric integration the integral vanishes, whereas dimensional regularization leads to $F_{\text{DREG}}(n) = i\pi^2 \left(k_2^\mu k_1^\nu - k_1 \cdot k_2 g^{\mu\nu}\right) + O(n-4)$. The integral has the same indeterminate behaviour of the former one: the value depends on the choice of the boundary. We can say that $F = J k_2^\mu k_1^\nu + J' g^{\mu\nu}$, with J and J' indeterminate constants. While in the (6.22) the tensor $k_2^\mu k_1^\nu$ has a well-defined finite coefficient, and we can tune the rest of the amplitude on it, in unitary gauge this coefficient is indeterminate, possibly divergent: we must then add another counterterm $\delta g_{0A} h_0 \left(\partial^\mu A_0^\nu\right)^2$ to absorb the divergence.

We have now *two counterterms* and we need *two renormalization conditions* to fix the arbitrariness. The Dyson subtraction alone (which means imposing gauge invariance) is not enough anymore. The result in (3.32) is still arbitrary, and allows the addition of whatever gauge invariant $k_2^\mu k_1^\nu - k_1 \cdot k_2\, g^{\mu\nu}$ term. The other condition could be, for example, the requirement of the validity of the Equivalence theorem [30–32] in the limit $m_W \to 0$, or the invariance of the amplitude in all gauges: both conditions fix the value of the amplitude in (3.32) to the standard result in (3.34). The indeterminacy is resolved; we also recover the independence of the amplitude on gauge choice and regulator choice.

Chapter 7

Conclusions

We have analyzed the computation of the amplitude $H \to \gamma\gamma$ by Gastmans *et al.* [1,2], to understand why it turns out to be different from the standard result of Refs. [3,4]. Integrals of the form

$$I_{\mu\nu} = \int d^4l \, \frac{g_{\mu\nu} l^2 - 4 l_\mu l_\nu}{\left(l^2 - M^2 + i\epsilon\right)^3} \tag{7.1}$$

are not well defined. We have provided some explicit examples within cutoff regularizations, obtaining different values by varying integration boundaries.

In 't Hooft-Feynman gauge and in a cutoff regularization scheme, see (6.22), we obtain

$$\mathcal{M}^{\mu\nu}_{\xi=1} = \frac{e^2 g}{(4\pi)^2 m_W} \left[-k_2^\mu k_1^\nu \left(2 + 3\tau^{-1} + 3\tau^{-1}\left(2 - \tau^{-1}\right) f(\tau)\right) + I g^{\mu\nu} \right] \tag{7.2}$$

where I is a constant which depends on the boundary shape. This makes it indeterminate as there is no *physical prescription* on the choice of the integration boundary shape. On the other hand, the use of a gauge invariant regularization scheme automatically provides the recipe on how to evaluate the integrals.

Since the term $k_2^\mu k_1^\nu$, in (7.1), has *only one* finite unambiguous coefficient, we are able to solve the indeterminacy by imposing gauge invariance at the end of the calculation.

However, we have shown that, in the unitary gauge, both the coefficients $k_2^\mu k_1^\nu$ and $g^{\mu\nu}$ are indeterminate in the sense of integration boundary shape dependency.

Imposing only one renormalization condition (like imposing gauge invariance by Dyson subtraction) *is not enough anymore*. Given the equivalence of R_ξ gauges with unitary gauge as $\xi \to \infty$, the problem we discuss is likely to be related to the exchange of this limit with integral sign for non Riemann-summable functions: the coefficient of $k_2^\mu k_1^\nu$ arises from highly divergent terms which do not appear at finite values of ξ.

Gastmans *et al.*'s expression in (3.32) is still ambiguous after Dyson subtraction, and allows the addition of whatever term of the form $k_2^\mu k_1^\nu - k_1 \cdot k_2\, g^{\mu\nu}$. This arbitrariness can be fixed by requiring the validity of the Equivalence theorem, or by imposing the equality of amplitudes in unitary and 't Hooft-Feynman gauges. In other words, we are able to add terms to (3.34) in order to match the standard result (3.32).

The combination of unitary gauge with a cutoff regularization scheme simply turns out to be non-predictive.

Appendix A

Some useful integrals

A.1 4-D and n-D integrals

We show how to evaluate integrals of the form

$$I = \int \frac{d^n l}{(2\pi)^n} \frac{(l^2)^\alpha}{(l^2 - M^2 + i\epsilon)^\beta} \tag{A.1}$$

The superficial degree of divergence is $n + 2\alpha - 2\beta$; if divergent, the integral has to be regularized in some way (for example, with dimensional regularization by taking the limit $n \to 4$ at the end of calculations, or with some cutoffs).

First, we can Wick-rotate the integral, so that $d^n l = i d^n l_E$ and $l^2 = -l_E^2$, and rescale $l_E \to l_E M$. The poles near the integration path are moved away, so we can take the limit $\epsilon \to 0$.

$$I = i\,(-1)^{\alpha+\beta}\, M^{n+2\alpha-2\beta} \int \frac{d^n l}{(2\pi)^n} \frac{(l^2)^\alpha}{(1 + l^2)^\beta} \tag{A.2}$$

In polar coordinates,

$$I = i\,(-1)^{\alpha+\beta}\, M^{n+2\alpha-2\beta} \frac{\Omega_n}{(2\pi)^n} \int_0^\infty dl\, l^{n-1} \frac{(l^2)^\alpha}{(1 + l^2)^\beta} \tag{A.3}$$

where $\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface of the unitary n-sphere. After some algebra,

$$
\begin{aligned}
I &= \frac{i\,(-1)^{\alpha+\beta}\,M^{n+2\alpha-2\beta}}{(4\pi)^{n/2}\,\Gamma(n/2)} \int_0^\infty dl^2 \frac{\left(l^2\right)^{\frac{n}{2}-1+\alpha}}{\left(1+l^2\right)^{\beta}} \\
&= \frac{i\,(-1)^{\alpha+\beta}\,M^{n+2\alpha-2\beta}}{(4\pi)^{n/2}\,\Gamma(n/2)\Gamma(\beta)} \int_0^\infty dy \int_0^\infty dl^2\, \left(l^2\right)^{\frac{n}{2}-1+\alpha} y^{\beta-1} \exp\left(-y\left(1+l^2\right)\right) \\
&\xrightarrow{yl^2\to l^2} \frac{i\,(-1)^{\alpha+\beta}\,M^{n+2\alpha-2\beta}}{(4\pi)^{n/2}\,\Gamma(n/2)\Gamma(\beta)} \int_0^\infty dy \int_0^\infty dl^2\, \left(l^2\right)^{\frac{n}{2}-1+\alpha} y^{\beta-\frac{n}{2}-\alpha-1} \exp\left(-y-l^2\right) \\
&= \frac{i\,(-1)^{\alpha+\beta}\,M^{n+2\alpha-2\beta}}{(4\pi)^{n/2}\,\Gamma(n/2)\Gamma(\beta)} \Gamma\left(\frac{n}{2}+\alpha\right) \Gamma\left(\beta-\frac{n}{2}-\alpha\right)
\end{aligned}
\tag{A.4}
$$

This result has to be read differently for different regularizations: in dimensionsional regularization we treat n as a continuos parameter and take the limit $n \to 4$; in a cutoff scheme, n is fixed but the divergent Gamma function has to be intended as a Gamma incomplete function.

The integral studied for the $H \to \gamma\gamma$ amplitude in (3.37) has $\alpha = 1, \beta = 3$; in dimensional regularization,

$$
\begin{aligned}
I &= \frac{iM^{n-4}}{(4\pi)^{n/2}\,\Gamma(n/2)2} \Gamma\left(\frac{n}{2}+1\right) \Gamma\left(2-\frac{n}{2}\right) \\
&\approx \frac{iM^{n-4}}{(4\pi)^{n/2}} \frac{n}{2\,(4-n)} + \text{finite terms}
\end{aligned}
\tag{A.5}
$$

When multiplied by $\frac{n-4}{n}$, it gets to

$$
\frac{n-4}{n} I = \frac{n-4}{n} \frac{iM^{n-4}}{(4\pi)^{n/2}} \frac{n}{2\,(4-n)} = \frac{-i}{2\,(4\pi)^2}
\tag{A.6}
$$

That is the result of (3.37), up to a factor $(2\pi)^4$.

A.2 Integrals over Feynman parameters

In equation (3.31) we found the following integral:

$$
I = \int_{\text{simplex}} dx_1 dx_2 \frac{1-2x_1x_2}{m_W^2 - x_1x_2m_H^2}
\tag{A.7}
$$

Firstly, we define the variable $\tau = \frac{m_H^2}{4m_W^2}$, and separate the fraction:

$$
\begin{aligned}
I =& \frac{1}{m_W^2} \int_{\text{simplex}} dx_1 dx_2 \frac{\frac{1}{2\tau} - 2x_1x_2 + 1 - \frac{1}{2\tau}}{1 - 4\tau\, x_1x_2} \\
=& \frac{1}{m_W^2} \int_{\text{simplex}} dx_1 dx_2 \frac{\frac{1}{2\tau} - 2x_1x_2 + 1 - \frac{1}{2\tau}}{1 - 4\tau\, x_1x_2} \\
=& \frac{1}{m_W^2} \left(\int_{\text{simplex}} dx_1 dx_2 \frac{1}{2\tau} + \left(1 - \tfrac{1}{2\tau}\right) \int_{\text{simplex}} dx_1 dx_2 \frac{1}{1 - 4\tau\, x_1 x_2} \right) \\
=& \frac{1}{m_W^2} \left(\frac{1}{4\tau} + \left(1 - \tfrac{1}{2\tau}\right) \int_{\text{simplex}} dx_1 dx_2 \frac{1}{1 - 4\tau\, x_1 x_2} \right)
\end{aligned} \tag{A.8}
$$

We focus on the integral

$$
\begin{aligned}
J =& \int_{\text{simplex}} dx_1 dx_2 \frac{1}{1 - 4\tau\, x_1 x_2} \\
=& \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{1 - 4\tau\, x_1 x_2} \\
=& \int_0^1 dx_1 \frac{-\log\left(1 - 4\tau\, x_1 \left(1 - x_1\right)\right)}{4\tau x_1}
\end{aligned} \tag{A.9}
$$

To solve the integral, we derive and integrate with respect to τ:

$$
\begin{aligned}
J =& \frac{1}{4\tau} \int_0^1 dx_1 \int_0^{\tau} d\tau' \frac{d}{d\tau'} \frac{-\log\left(1 - 4\tau'\, x_1 \left(1 - x_1\right)\right)}{x_1} \\
=& \frac{1}{4\tau} \int_0^{\tau} d\tau' \int_0^1 dx_1 \frac{4\left(1 - x_1\right)}{1 - 4\tau'\, x_1 \left(1 - x_1\right)}
\end{aligned} \tag{A.10}
$$

We explore the integral in dx_1. We shift $x_1 \to x_1 + \frac{1}{2}$:

$$
\int_0^1 dx_1 \frac{4\left(1 - x_1\right)}{1 - 4\tau'\, x_1 \left(1 - x_1\right)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 \frac{4\left(\frac{1}{2} - x_1\right)}{1 - 4\tau' \left(\frac{1}{4} - x_1^2\right)} \tag{A.11}
$$

The domain and the denominator are symmetric to $x_1 \to -x_1$, so we can drop the odd term in the numerator.

$$\begin{aligned}
\int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 \frac{2}{1-\tau' + 4\tau' x_1^2} &= \frac{1}{1-\tau'} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 \frac{2}{1+\left(2\frac{\sqrt{\tau'}}{\sqrt{1-\tau'}}\, x_1\right)^2} \\
&= \frac{1}{\sqrt{\tau'\left(1-\tau'\right)}} \arctan 2\frac{\sqrt{\tau'}}{\sqrt{1-\tau'}} x_1 \Bigg|_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= 2\frac{1}{\sqrt{\tau'\left(1-\tau'\right)}} \arctan \frac{\sqrt{\tau'}}{\sqrt{1-\tau'}} = 2\frac{1}{\sqrt{\tau'\left(1-\tau'\right)}} \arcsin \sqrt{\tau'}
\end{aligned} \tag{A.12}$$

where in the last line we used $\arctan x = \arcsin \frac{x}{\sqrt{1-x^2}}$. Finally, we can integrate with respect to τ':

$$\begin{aligned}
J &= \frac{1}{2\tau} \int_0^\tau d\tau' \frac{1}{\sqrt{\tau'\left(1-\tau'\right)}} \arcsin \sqrt{\tau'} = \frac{1}{\tau} \int_0^\tau \arcsin \sqrt{\tau'}\, d\left(\arcsin \sqrt{\tau'}\right) \\
&= \frac{\arcsin^2 \sqrt{\tau}}{2\tau}
\end{aligned} \tag{A.13}$$

And so

$$I = \frac{1}{m_W^2} \left(\frac{1}{4\tau} + \left(1 - \tfrac{1}{2\tau}\right) \frac{\arcsin^2 \sqrt{\tau}}{2\tau} \right) = \frac{1}{4m_W^2} \left(\tau^{-1} + \left(2\tau^{-1} - \tau^{-2}\right) \arcsin^2 \sqrt{\tau}\right) \tag{A.14}$$

The arcsin can be analitically continued for $\tau > 1$, giving:

$$I = \frac{1}{4m_W^2} \left(\tau^{-1} + \left(2\tau^{-1} - \tau^{-2}\right) f(\tau)\right) \tag{A.15}$$

with

$$f(\tau) = \begin{cases} \arcsin^2(\sqrt{\tau}) & \text{for} \quad \tau \le 1 \\ -\dfrac{1}{4} \left[\ln \dfrac{1+\sqrt{1-\tau^{-1}}}{1-\sqrt{1-\tau^{-1}}} - i\pi\right]^2 & \text{for} \quad \tau > 1 \end{cases} \tag{A.16}$$

In general, we have:

$$\int_{\text{simplex}} dx_1 dx_2 \frac{a - 2bx_1x_2}{m_W^2 - x_1x_2m_H^2} = \frac{1}{4m_W^2} \left(b\tau^{-1} + \left(2a\tau^{-1} - b\tau^{-2}\right) f(\tau)\right) \tag{A.17}$$

Appendix B

Passarino-Veltman reduction

B.1 Dimensional regularization

The Passarino-Veltman scheme [33, 34] is an efficient way allowing one to express one loop Feynman diagram as a sum of basic scalar integrals times some coefficients depending only on external kinematical quantities.

We define the general one-loop tensor integral (see Fig. 5.3) in dimensional regularization scheme as

$$T^N_{\mu_1 \ldots \mu_P}(p_1, \ldots, p_{N-1}, m_0, \ldots, m_{N-1}) = \frac{(2\pi\mu)^{4-n}}{i\pi^2} \int d^n q \frac{q_{\mu_1} \cdots q_{\mu_P}}{D_0 D_1 \cdots D_{N-1}} \tag{B.1}$$

with the denominator factors

$$D_0 = q^2 - m_0^2 + i\epsilon, \qquad D_i = (q + p_i)^2 - m_i^2 + i\epsilon, \qquad i = 1, \ldots, N-1, \tag{B.2}$$

originating from the propagators in the Feynman diagram. Furthermore we introduce

$$p_{i0} = p_i \quad \text{and} \quad p_{ij} = p_i - p_j. \tag{B.3}$$

Evidently the tensor integrals are invariant under arbitrary permutations of the propagators D_i, $i \neq 0$ and totally symmetric in the Lorentz indices μ_k. $i\epsilon$ is an infinitesimal imaginary part which is needed to regulate singularities of the integrand. Its specific choice ensures causality. After integration it determines the correct imaginary parts of the logarithms and dilogarithms.

The parameter μ has mass dimension and serves to keep the dimension of the integrals fixed for varying n. Conventionally T^N is denoted by the Nth character of the alphabet, i.e. $T^1 \equiv A$, $T^2 \equiv B$, $\ldots$, and the scalar integrals carry an index 0.

It can be shown that every loop integral can be reduced to a combination of A_0, B_0, C_0 and D_0 times functions of external momenta. Lorentz covariance of the integrals allows to decompose the tensor integrals into tensors constructed from the external momenta p_i, and the metric tensor $g_{\mu\nu}$ with totally symmetric coefficient functions $T^N_{i_1 \ldots i_P}$. For example,

$$\begin{aligned}
B_\mu &= p_{1\mu} B_1 \\
B_{\mu\nu} &= g_{\mu\nu} B_{00} + p_{1\mu} p_{1\nu} B_{11} \\
C_\mu &= p_{1\mu} C_1 + p_{2\nu} C_2 \\
C_{\mu\nu} &= g_{\mu\nu} C_{00} + p_{1\mu} p_{1\nu} C_{11} + p_{1\mu} p_{2\nu} C_{12} + p_{2\mu} p_{1\nu} C_{21} + p_{2\mu} p_{2\nu} C_{22} \\
&\ldots
\end{aligned} \tag{B.4}$$

The relations between tensor integrals and scalar integrals can be obtained by saturating both sides with $g_{\mu\nu}$ or with external momenta, and solving with respect to the scalar integrals. For example, we derive

$$\begin{aligned}
p_1^\mu C_\mu &= p_1^2 C_1 + (p_1 \cdot p_2)\, C_2 \\
p_2^\mu C_\mu &= (p_1 \cdot p_2)\, C_1 + p_2^2 C_2
\end{aligned} \tag{B.5}$$

therefore

$$\begin{aligned}
p_1^\mu C_\mu &= C \int d^n q \frac{p_1 \cdot q}{\left(q^2 - m_0^2 + i\epsilon\right)\left(\left(q+p_1\right)^2 - m_1^2 + i\epsilon\right)\left(\left(q+p_2\right)^2 - m_2^2 + i\epsilon\right)} \\
&= \frac{C}{2} \int d^n q \frac{\left(p_1 + q\right)^2 - p_1^2 - q^2}{\left(q^2 - m_0^2 + i\epsilon\right)\left(\left(q+p_1\right)^2 - m_1^2 + i\epsilon\right)\left(\left(q+p_2\right)^2 - m_2^2 + i\epsilon\right)} \\
&= \frac{1}{2}\Big[B_0\left(p_2, m_0, m_2\right) - B_0\left(p_{21}, m_1, m_2\right) \\
&\qquad - \left(m_0^2 - m_1^2 + p_1^2\right) C_0\left(p_1, p_2, m_0, m_1, m_2\right)\Big]
\end{aligned} \tag{B.6}$$

and the same for $p_2^\mu C_\mu$. By solving this linear system in C_1, C_2, we can obtain the final expressions. We remark that, in order to get the $B_0\left(p_{21}, m_1, m_2\right)$, we have shifted the integral $q \to q - p_1$. In dimensional regularization this can be done in any case, whereas in gauge-breaking schemes one should be more careful.

We report some explicit expressions for the scalar integrals:

$$\begin{aligned} A_0\,(m) &= -m^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \Gamma(\epsilon - 1) \\ &= m^2 \left(\frac{1}{\tilde{\epsilon}} - \log\frac{m^2}{\mu^2} + 1\right) + O\,(4-n) \qquad \text{(B.7a)} \\ B_0(p_{10}, m_0, m_1) &= \frac{1}{\tilde{\epsilon}} - \int_0^1 dx \log\frac{[p_{10}^2 x^2 - x(p_{10}^2 - m_0^2 + m_1^2) + m_1^2 - i\epsilon]}{\mu^2} \\ &\quad + O\,(4-n) \qquad \text{(B.7b)} \\ C_0(M^2, 0, 0, m^2, m^2, m^2) &= -\frac{2}{M^2} f\left(\frac{M^2}{4m^2}\right) \qquad \text{(B.7c)} \end{aligned}$$

where $\frac{1}{\epsilon} = \frac{2}{4-n}$, $\frac{1}{\tilde{\epsilon}} = \frac{1}{\epsilon} - \gamma_E + \log 4\pi$, and f is defined in (A.16).

B.2 Cutoff regularization

If we use a cutoff scheme, the reduction scheme described in the previous section can still work, but we must pay attention to the surface terms we can generate when we shift integrals with more-than-logarithmic divergences (for the details, see [9]).

We just recall some results which differ from dimensional regularized integrals.

$$A_0\,(0) = -i\pi^2\,\Lambda^2 \qquad \text{(B.8a)}$$

$$A_0\left(m_0^2\right) = i\pi^2 m_0^2 \left(\ln\left(\frac{\Lambda^2}{m_0^2}\right) - \frac{\Lambda^2}{m_0^2}\right) \qquad \text{(B.8b)}$$

$$B_0\left(p_1^2, m_0^2, m_1^2\right) = i\pi^2 \left(\ln\left(\frac{\Lambda^2}{p_1^2}\right) + 1 + \sum_{i=1}^{2}\left[\gamma_i \ln\left(\frac{\gamma_i - 1}{\gamma_i}\right) - \ln\left(\gamma_i - 1\right)\right]\right) \qquad \text{(B.8c)}$$

$$\text{with} \quad \gamma_{1,2} = \frac{p_1^2 - m_1^2 + m_0^2 \pm \sqrt{\left(p_1^2 - m_1^2 + m_0^2\right)^2 - 4p_1^2 m_0^2}}{2p_1^2} \qquad \text{(B.8d)}$$

We see that $A_0(0)$ is quadratically divergent, whereas it vanishes in dimensional regularization[1]. The results for the finite C_0 are the same in all regularizations, being the value of a finite integral univocal.

[1]This happens because the tadpole integral $\int \frac{d^n q}{q^\alpha}$ identically vanishes because of the cancellations of infrared and ultraviolet divergences.

Appendix C

$H \to \gamma\gamma$ in general R_ξ gauge

We mean to show how, in dimensional regularization scheme, the ξ dependence cancels out, leading to the same result as in unitary gauge. We have shown the diagrams in Fig. 5.1.

Following [7], we divide the W boson propagator into two parts

$$\begin{aligned}\Delta^{\mu\nu} &= \frac{-i}{q^2 - m_W^2}\left(g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2 - \xi m_W^2}\right) \\ &= \frac{-i}{q^2 - m_W^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2}\right) + \frac{-i}{q^2 - \xi m_W^2}\frac{q^\mu q^\nu}{m_W^2}\end{aligned} \tag{C.1}$$

The first term on the right-hand side is a propagator in the unitary gauge. The second term has a $q^2 - \xi m_W^2$ in the denominator, and thus can be combined with Goldstone boson and ghost propagators that appear in other diagrams, to simplify the calculation.

Using this method, the diagrams with W propagators are divided into several parts. For example, the diagram $\mathcal{M}_{WWW}$ has 8 pieces. We denote them by $\mathcal{M}_{ijk}$ where $i, j, k = 1, 2$ according to which term on the right-hand side of (C.1) the W-propagator takes.

$$\mathcal{M}_{WWW} = \mathcal{M}_{111} + \mathcal{M}_{112} + \mathcal{M}_{121} + \mathcal{M}_{211} + \mathcal{M}_{122} + \mathcal{M}_{212} + \mathcal{M}_{221} + \mathcal{M}_{222} \tag{C.2}$$

with

$$\mathcal{M}_{111} = \int \frac{d^np}{(2\pi)^n} 2V^{\alpha\beta\gamma\delta\lambda\rho\mu\nu} \frac{g_{\alpha\gamma} - \frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - m_W^2} \frac{g_{\lambda\rho} - \frac{(p-k_1)_\lambda (p-k_1)_\rho}{m_W^2}}{(p-k_1)^2 - m_W^2} \frac{g_{\delta\beta} - \frac{(p-k_1-k_2)_\delta (p-k_1-k_2)_\beta}{m_W^2}}{(p-k_1-k_2)^2 - m_W^2} \tag{C.3}$$

$$\mathcal{M}_{112} = \int \frac{d^np}{(2\pi)^n} 2V^{\alpha\beta\gamma\delta\lambda\rho\mu\nu} \frac{g_{\alpha\gamma} - \frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - m_W^2} \frac{g_{\lambda\rho} - \frac{(p-k_1)_\lambda (p-k_1)_\rho}{m_W^2}}{(p-k_1)^2 - m_W^2} \frac{\frac{(p-k_1-k_2)_\delta (p-k_1-k_2)_\beta}{m_W^2}}{(p-k_1-k_2)^2 - \xi m_W^2} \tag{C.4}$$

$$\mathcal{M}_{121} = \int \frac{d^np}{(2\pi)^n} 2V^{\alpha\beta\gamma\delta\lambda\rho\mu\nu} \frac{g_{\alpha\gamma} - \frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - m_W^2} \frac{\frac{(p-k_1)_\lambda (p-k_1)_\rho}{m_W^2}}{(p-k_1)^2 - \xi m_W^2} \frac{g_{\delta\beta} - \frac{(p-k_1-k_2)_\delta (p-k_1-k_2)_\beta}{m_W^2}}{(p-k_1-k_2)^2 - m_W^2} \tag{C.5}$$

$$\mathcal{M}_{211} = \int \frac{d^np}{(2\pi)^n} 2V^{\alpha\beta\gamma\delta\lambda\rho\mu\nu} \frac{\frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - \xi m_W^2} \frac{g_{\lambda\rho} - \frac{(p-k_1)_\lambda (p-k_1)_\rho}{m_W^2}}{(p-k_1)^2 - m_W^2} \frac{g_{\delta\beta} - \frac{(p-k_1-k_2)_\delta (p-k_1-k_2)_\beta}{m_W^2}}{(p-k_1-k_2)^2 - m_W^2} \tag{C.6}$$

$$\ldots$$

$$\mathcal{M}_{222} = \int \frac{d^np}{(2\pi)^n} 2V^{\alpha\beta\gamma\delta\lambda\rho\mu\nu} \frac{\frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - \xi m_W^2} \frac{\frac{(p-k_1)_\lambda (p-k_1)_\rho}{m_W^2}}{(p-k_1)^2 - \xi m_W^2} \frac{\frac{(p-k_1-k_2)_\delta (p-k_1-k_2)_\beta}{m_W^2}}{(p-k_1-k_2)^2 - \xi m_W^2} \tag{C.7}$$

where

$$\begin{aligned} V^{\alpha\beta\gamma\delta\lambda\rho\mu\nu} &= -ie^2 g m_W g^{\alpha\beta} \left[(2p-k_1)^\mu g^{\gamma\lambda} - (p+k_1)^\lambda g^{\mu\gamma} - (p-2k_1)^\gamma g^{\mu\lambda}\right] \\ &\times \left[-(p-k_1+k_2)^\delta g^{\nu\rho} - (p-k_1-2k_2)^\rho g^{\nu\delta} + (2p-2k_1-k_2)^\nu g^{\rho\delta}\right] \end{aligned} \tag{C.8}$$

denotes the contribution from the vertices. A factor of 2 is included to take the crossed diagrams into account. This diagram can be obtained by $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$. Since we are only interested in terms that are proportional to either $g^{\mu\nu}$ or $k_2^\mu k_1^\nu$, the contribution from this diagram is the same.

There are also diagrams with both W and Goldstone boson propagators. We use the same notation, but with the subscript 0 to denote a Goldstone boson propagator:

$$\begin{aligned}
\mathcal{M}_{WW\phi} &= \mathcal{M}_{110} + \mathcal{M}_{120} + \mathcal{M}_{210} + \mathcal{M}_{220} \\
\mathcal{M}_{W\phi\phi} &= \mathcal{M}_{100} + \mathcal{M}_{200} \\
\mathcal{M}_{W\phi W} &= \mathcal{M}_{101} + \mathcal{M}_{102} + \mathcal{M}_{201} + \mathcal{M}_{202} \\
\mathcal{M}_{\phi W\phi} &= \mathcal{M}_{010} + \mathcal{M}_{020} \\
\mathcal{M}_{\phi\phi\phi} &= \mathcal{M}_{000}
\end{aligned} \tag{C.9}$$

For $\mathcal{M}_{WW\phi}$, we have

$$\mathcal{M}_{110} = \int \frac{d^n p}{(2\pi)^n} (-4) V'^{\alpha\gamma\lambda\rho\mu\nu} \frac{g_{\alpha\gamma} - \frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - m_W^2} \frac{g_{\lambda\rho} - \frac{(p-k_1)_\lambda (p-k_1)_\rho}{m_W^2}}{(p-k_1)^2 - m_W^2} \frac{1}{(p-k_1-k_2)^2 - \xi m_W^2} \tag{C.10}$$

$$\mathcal{M}_{120} = \int \frac{d^n p}{(2\pi)^n} (-4) V'^{\alpha\gamma\lambda\rho\mu\nu} \frac{g_{\alpha\gamma} - \frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - m_W^2} \frac{\frac{(p-k_1)_\lambda (p-k_1)_\rho}{m_W^2}}{(p-k_1)^2 - \xi m_W^2} \frac{1}{(p-k_1-k_2)^2 - \xi m_W^2} \tag{C.11}$$

$$\mathcal{M}_{210} = \int \frac{d^n p}{(2\pi)^n} (-4) V'^{\alpha\gamma\lambda\rho\mu\nu} \frac{\frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - \xi m_W^2} \frac{g_{\lambda\rho} - \frac{(p-k_1)_\lambda (p-k_1)_\rho}{m_W^2}}{(p-k_1)^2 - m_W^2} \frac{1}{(p-k_1-k_2)^2 - \xi m_W^2} \tag{C.12}$$

$$\mathcal{M}_{220} = \int \frac{d^n p}{(2\pi)^n} (-4) V'^{\alpha\gamma\lambda\rho\mu\nu} \frac{\frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - \xi m_W^2} \frac{\frac{(p-k_1)_\lambda (p-k_1)_\rho}{m_W^2}}{(p-k_1)^2 - \xi m_W^2} \frac{1}{(p-k_1-k_2)^2 - \xi m_W^2} \tag{C.13}$$

and

$$\begin{aligned}
V'^{\alpha\gamma\lambda\rho\mu\nu} = {} & i\frac{1}{2} e^2 g m_W (p - 2k_1 - 2k_2)^\alpha \\
& \times \left[(2p - k_1)^\mu g^{\gamma\lambda} - (p + k_1)^\lambda g^{\mu\gamma} - (p - 2k_1)^\gamma g^{\mu\lambda} \right] g^{\nu\rho}
\end{aligned} \tag{C.14}$$

Similarly for $\mathcal{M}_{W\phi\phi}$, $\mathcal{M}_{W\phi W}$, $\mathcal{M}_{\phi W\phi}$ and $\mathcal{M}_{\phi\phi\phi}$. These terms all include a factor of 2 from exchanging the external photons. $\mathcal{M}_{WW\phi}$ and $\mathcal{M}_{W\phi\phi}$ have another factor of 2, due to equal contributions from the diagrams $\mathcal{M}_{\phi WW}$ and $\mathcal{M}_{\phi\phi W}$.

Diagrams $\mathcal{M}_{WW}$, $\mathcal{M}_{W\phi}$ and $\mathcal{M}_{\phi\phi}$ only have two propagators. We denote them by

$$\mathcal{M}_{WW} = \mathcal{M}_{11} + \mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{22} \tag{C.15}$$

$$\mathcal{M}_{W\phi} = \mathcal{M}_{10} + \mathcal{M}_{20} \tag{C.16}$$

$$\mathcal{M}_{\phi\phi} = \mathcal{M}_{00} \tag{C.17}$$

The notation is similar to before. For example,

$$\mathcal{M}_{11} = \int \frac{d^n p}{(2\pi)^n} ie^2 g m_W g^{\alpha\beta} S^{\mu\nu,\gamma\delta} \frac{g_{\alpha\gamma} - \frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - m_W^2} \frac{g_{\delta\beta} - \frac{(p-k_1-k_2)_\delta (p-k_1-k_2)_\beta}{m_W^2}}{(p-k_1-k_2)^2 - m_W^2} \tag{C.18}$$

$$\mathcal{M}_{12} = \int \frac{d^n p}{(2\pi)^n} ie^2 g m_W g^{\alpha\beta} S^{\mu\nu,\gamma\delta} \frac{g_{\alpha\gamma} - \frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - m_W^2} \frac{\frac{(p-k_1-k_2)_\delta (p-k_1-k_2)_\beta}{m_W^2}}{(p-k_1-k_2)^2 - \xi m_W^2} \tag{C.19}$$

$$\mathcal{M}_{21} = \int \frac{d^n p}{(2\pi)^n} ie^2 g m_W g^{\alpha\beta} S^{\mu\nu,\gamma\delta} \frac{\frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - \xi m_W^2} \frac{g_{\delta\beta} - \frac{(p-k_1-k_2)_\delta (p-k_1-k_2)_\beta}{m_W^2}}{(p-k_1-k_2)^2 - m_W^2} \tag{C.20}$$

$$\mathcal{M}_{22} = \int \frac{d^n p}{(2\pi)^n} ie^2 g m_W g^{\alpha\beta} S^{\mu\nu,\gamma\delta} \frac{\frac{p_\alpha p_\gamma}{m_W^2}}{p^2 - \xi m_W^2} \frac{\frac{(p-k_1-k_2)_\delta (p-k_1-k_2)_\beta}{m_W^2}}{(p-k_1-k_2)^2 - \xi m_W^2} \tag{C.21}$$

and $S_{\mu\nu,\gamma\delta} = 2g_{\mu\nu}g_{\gamma\delta} - g_{\mu\gamma}g_{\nu\delta} - g_{\mu\delta}g_{\nu\gamma}$. Similarly for $\mathcal{M}_d$ and $\mathcal{M}_i$.

Lastly, there is a ghost loop diagram:

$$\mathcal{M}_{\eta^+} = \int \frac{d^n p}{(2\pi)^n} 2ie^2 g m_W \xi \frac{(p-k_1)^\mu (p-k_1-k_2)^\nu}{(p^2 - \xi m_W^2)[(p-k_1)^2 - \xi m_W^2][(p-k_1-k_2)^2 - \xi m_W^2]} \tag{C.22}$$

$\mathcal{M}_{\eta^+}$ has a factor of -1 from the ghost loop. Diagrams $\mathcal{M}_{W\phi}$ and $\mathcal{M}_{\eta^+}$ contain a factor of 4 from exchanging the external photons and from charge conjugation (which transform them into the equal diagrams $\mathcal{M}_{\phi W}$ and $\mathcal{M}_{\eta^-}$).

Some of these terms vanish:

$$\mathcal{M}_{122} = \mathcal{M}_{221} = \mathcal{M}_{222} = \mathcal{M}_{220} = 0 \tag{C.23}$$

In the remaining terms, certain combinations will give simple results. For example, the contribution from pure Goldstone boson loops is gauge invariant:

$$
\begin{aligned}
&\mathcal{M}_{000} + \mathcal{M}_{00} = \mathcal{M}_{\phi\phi\phi} + \mathcal{M}_{\phi\phi} \\
=& - i\frac{e^2 g m_H^2}{m_W} \int \frac{d^n p}{(2\pi)^n} \left[\frac{4p^\mu (p-k_1)^\nu}{(p^2 - \xi m_W^2)[(p-k_1)^2 - \xi m_W^2][(p-k_1-k_2)^2 - \xi m_W^2]} \right. \\
&\left. - \frac{g^{\mu\nu}}{(p^2 - \xi m_W^2)[(p-k_1-k_2)^2 - \xi m_W^2]} \right] \\
=& \frac{2e^2 g}{(4\pi)^2 m_W} \left[1 + 2\xi m_W^2 C_0(m_H^2, 0, 0, \xi m_W^2, \xi m_W^2, \xi m_W^2) \right] \left[(k_1 \cdot k_2) g^{\mu\nu} - k_2^\mu k_1^\nu \right]
\end{aligned}
\tag{C.24}
$$

All the remaining terms with no 1 in the subscript should be combined. We find

$$
\begin{aligned}
&\mathcal{M}_{20} + \mathcal{M}_{200} + \mathcal{M}_{202} + \mathcal{M}_{020} + \mathcal{M}_{\eta^+} \\
=& i\frac{e^2 g}{m_W} \int \frac{d^n p}{(2\pi)^n} \left[\frac{4m_H^2 p^\mu (p-k_1)^\nu}{(p^2 - \xi m_W^2)[(p-k_1)^2 - \xi m_W^2][(p-k_1-k_2)^2 - \xi m_W^2]} \right. \\
&\left. + \frac{3p^\mu (p-k_1)^\nu}{[(p-k_1)^2 - \xi m_W^2][(p-k_1-k_2)^2 - \xi m_W^2]} - \frac{3p^\mu (p-k_1)^\nu}{(p^2 - \xi m_W^2)[(p-k_1)^2 - \xi m_W^2]} \right]
\end{aligned}
\tag{C.25}
$$

The last two terms cancel each other under $p_1 \leftrightarrow p_2$, $\mu \leftrightarrow \nu$ and momentum shifting. The first term then gives

$$
\begin{aligned}
&\mathcal{M}_{20} + \mathcal{M}_{200} + \mathcal{M}_{202} + \mathcal{M}_{020} + \mathcal{M}_{\eta^+} \\
=& - \frac{e^2 g}{(4\pi)^2 m_W} \left\{ 2 \left[1 + 2\xi m_W^2 C_0(m_H^2, 0, 0, \xi m_W^2, \xi m_W^2, \xi m_W^2) \right] \left[(k_1 \cdot k_2) g^{\mu\nu} - k_2^\mu k_1^\nu \right] \right. \\
&\left. + m_H^2 B_0(m_H^2, \xi m_W^2, \xi m_W^2) g^{\mu\nu} \right\}
\end{aligned}
\tag{C.26}
$$

The first term on the right-hand side cancels the contribution from $\mathcal{M}_{000}$ and $\mathcal{M}_{00}$. The second term with a B_0 function is cancelled by $\mathcal{M}_{22} + \mathcal{M}_{212} + \mathcal{M}_{210} + \mathcal{M}_{010}$.

In fact,

$$
\begin{aligned}
&\mathcal{M}_{22} + \mathcal{M}_{212} + \mathcal{M}_{210} + \mathcal{M}_{010} \\
=& - i\frac{e^2 g}{m_W^3} \int \frac{d^n p}{(2\pi)^n} \left[\frac{1}{2} \left(p^\mu k_1^\nu - k_2^\mu p^\nu\right) \left(\frac{1}{p^2 - \xi m_W^2} + \frac{1}{(p-k_1-k_2)^2 - \xi m_W^2} \right) \right. \\
&- \left(p^\mu k_1^\nu - p^\mu p^\nu\right) \frac{m_W^2}{(p-k_1) - m_W^2} \left(\frac{1}{p^2 - \xi m_W^2} - \frac{1}{(p-k_1-k_2)^2 - \xi m_W^2} \right) \\
&+ \frac{\left(\xi m_W^2 - \frac{1}{2} m_H^2\right)\left(p^\mu k_1^\nu - k_2^\mu p^\nu\right)}{(p^2 - \xi m_W^2)[(p-k_1-k_2)^2 - \xi m_W^2]} \\
&-p \cdot (k_1 - k_2) g^{\mu\nu} \left(\frac{1}{p^2 - \xi m_W^2} + \frac{1}{(p-k_1-k_2)^2 - \xi m_W^2} \right) \\
&- \left((1-\xi) m_W^2 + \frac{1}{2} m_H^2 \right) g^{\mu\nu} \left(\frac{1}{p^2 - \xi m_W^2} - \frac{1}{(p-k_1-k_2)^2 - \xi m_W^2} \right) \\
&- \frac{m_W^4 g^{\mu\nu}}{(p-k_1) - m_W^2} \left(\frac{1}{p^2 - \xi m_W^2} - \frac{1}{(p-k_1-k_2)^2 - \xi m_W^2} \right) \\
&\left. + \frac{m_H^2 \left((1-\xi) m_W^2 + \frac{1}{2} m_H^2\right) + (4\xi m_W^2 - 2m_H^2) p \cdot k_2}{(p^2 - \xi m_W^2)[(p-k_1-k_2)^2 - \xi m_W^2]} g^{\mu\nu} \right]
\end{aligned}
\tag{C.27}
$$

It's not hard to see that under $k_1 \leftrightarrow k_2$, $\mu \leftrightarrow \nu$ and momentum shifting, all terms except the last term cancel out. We have

$$
\mathcal{M}_{22} + \mathcal{M}_{212} + \mathcal{M}_{210} + \mathcal{M}_{010} = \frac{e^2 g}{(4\pi)^2 m_W} m_H^2 B_0(m_H^2, \xi m_W^2, \xi m_W^2) g^{\mu\nu} \tag{C.28}
$$

All the remaining $\mathcal{M}$s should cancel. We find

$$
\mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{112} + \mathcal{M}_{211} + \mathcal{M}_{110} + \mathcal{M}_{10} = -(\mathcal{M}_{121} + \mathcal{M}_{101}) \tag{C.29}
$$

and

$$
\mathcal{M}_{120} + \mathcal{M}_{100} = -2\mathcal{M}_{102} = -2\mathcal{M}_{201} \tag{C.30}
$$

These all add up to zero, as expected. Thus we see that all terms except $\mathcal{M}_{11} + \mathcal{M}_{111}$ are cancelled. Thus, the unphysical dependence on ξ disappears, and we obtain the same result as in unitary gauge.

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