Written test of Advanced Quantum Mechanics

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Exam time: 2 hours. You can use the Clebsch-Gordan sheet by PDG.

EXERCISE 1

A particle of mass m and spin 1/2 moves in 3D space according to the following Hamiltonian:

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 + \frac{\alpha}{\hbar}\boldsymbol{J}^2 \tag{1}$$

with $\alpha \ll \omega$.

(i) Calculate energies and degenerations of the first 3 levels;

(ii) Consider a particle in the state

$$|\psi\rangle = AR_1(r)\cos\theta\,|+\rangle \ , \tag{2}$$

where $R_1(1)$ is the appropriate radial eigenfunction of the harmonic oscillator with $n = 1, l = 1$, and is normalized as $\int_0^\infty |R_1(r)|^2 r^2 dr = 1$. Calculate A so that the state is normalized, and determine the possible outcome of a measurement of H with the respective probabilities.

(iii) Calculate the time evolution of $|\psi\rangle$, and the mean values of L_z, $\boldsymbol{J}^2$, $\boldsymbol{L}^2$. Do they all depend on time? Why?

EXERCISE 2

Two identical particles of spin 1/2 are vinculated to a spherical surface of radius R. The dynamics are given by the following Hamiltonian:

$$H = \frac{\vec{L}_1^2}{2mR^2} + \frac{\vec{L}_2^2}{2mR^2} + \frac{\alpha}{mR^2}\vec{L}_1\cdot\vec{S}_1 + \frac{\alpha}{mR^2}\vec{L}_2\cdot\vec{S}_2 \tag{3}$$

where $0 < \alpha \ll 1$. Note that the system is not in the center of mass frame.

1. Calculate the ground and the first excited state

(*Hint:* combine each $L_{1,2}$ and $S_{1,2}$ into $J_{1,2}$, and then antisymmetrize)